AN INTRODUCTION TO THE MATHEMATICAL THEORY OF HEAT CONDUCTION

WITH ENGINEERING AND GEOLOGICAL APPLICATIONS

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MATHEMATICAL THEORY OF HEAT CONDUCTION

CHAPTER I

INTRODUCTION

1. Historical. The mathematical theory of heat conduction, historically speaking, is due principally to Jean Baptiste Joseph Fourier (1768–1830) and was set forth by him in his *Théorie analytique de la Chaleur.* While Lambert, Biot, and others had developed some more or less correct ideas on the subject, it was Fourier who first brought order out of the confusion in which the experimental physicists had left the subject. Professor Tait has said of his work: "Its exquisitely original methods have been the source of inspiration of some of the greatest mathematicians; and the mere application of one of its simplest portions to the conduction of electricity has made the name of Ohm famous," while Lord Kelvin remarks: * "Returning to the conduction of heat, we have first to say that the theory of it was discovered by Fourier, and given to the world through the French Academy in his *Théorie analytique de la Chaleur*, with solutions of problems naturally arising from it, of which it is difficult to say whether their uniquely original quality, or their transcendently intense mathematical interest, or their perennially important instructiveness for physical science, is most to be praised."

While Fourier treated a large number of cases, including most of those we shall have occasion to consider, his work was

* Article on Heat, *Encyclopedia Britannica*, ninth edition.

extended and applied to more complicated problems by his contemporaries Laplace and Poisson, and later by a number of others, including Lamé, Sir W. Thomson (Lord Kelvin), and Riemann, to which latter writer all students of the subject should feel indebted for the very readable form in which he has put much of Fourier's work.

2. Definitions. When different parts of a solid body are at different temperatures, heat flows from the hotter to the colder portions by a process of transference — probably from molecule to molecule — known as conduction. The rate at which heat will be so transferred has been found by experiment to depend on a number of conditions which we shall now consider.

To fix the ideas, imagine in a body two parallel planes, or laminæ, of area A and distance apart x, over each of which the temperature is constant, being θ_1 in one case and θ_2 in the other. Heat will then flow from the hotter of these isothermal surfaces to the cooler, and the quantity Q which will be conducted in time t will be given by

$$Q = k\frac{\theta_1 - \theta_2}{x}At, \tag{1}$$

where k is a constant for any given material known as the *thermal conductivity* of the substance. It is then numerically equal to the quantity of heat which flows in unit time through unit area of a plate of unit thickness having unit difference of temperature between its faces.

The limiting value of $\frac{\theta_1 - \theta_2}{x}$, or $\frac{\partial\theta}{\partial x}$, is known as the *temperature gradient* at any point. If due attention is paid to sign, we see that if $\frac{\partial\theta}{\partial x}$ is taken in the direction of the flow of heat, it is intrinsically negative. Hence we may write for the rate at which heat is transferred across an isothermal surface, per unit area,

$$-k\frac{\partial\theta}{\partial x}. \tag{2}$$

This is called the flux of heat across the surface at that point and may be denoted by W. If, instead of an isothermal surface,

PREFACE

For some years past, one of the authors has given a course in this subject at the University of Wisconsin, and has felt keenly the need of a suitable text. The present volume has been written primarily to meet this need, and in the hope that it might stimulate the more extensive study which the subject deserves; for the theory of heat conduction is of importance, not only intrinsically but also because its broad bearing and the generality of its methods of analysis make it one of the best introductions to more advanced mathematical physics.

The aim of the authors has been twofold: They have attempted, in the first place, to develop the subject with special reference to the needs of the student who has neither time nor mathematical preparation to pursue the study at great length. To this end fewer types of problems are handled than in the larger treatises, and less stress has been placed on purely mathematical derivations such as uniqueness, existence, and convergence theorems.

The second aim has been to point out more clearly and specifically than apparently has been done before, the many applications of which the results are susceptible; for in its practical bearing this field is second to no other in mathematical physics. This feature invariably awakens and holds the interest of the student who feels, all too frequently, that much of his previous mathematical training has been devoid of application.

It is hoped also that in this respect the subject matter may be of interest to the engineer, for the authors have attempted to select applications with special reference to their technical importance, and in furtherance of this idea have sought and received suggestions from engineers in many lines of work. While many of these applications have doubtless only a small practical bearing and serve chiefly to illustrate the theory, it is not impossible

that the results in some cases, for example, the "theory of the fire-proof wall," may be found worthy of note. The same may be said of the geological problems.

While a number of solutions are here presented for the first time, it is believed, no originality can be claimed for the underlying mathematical theory, which dates back, of course, to the time of Fourier. The authors are glad to acknowledge their indebtedness to Byerly's *Fourier's Series and Spherical Harmonics*, and to Carslaw's *Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat;* also, it is hardly necessary to add, to Riemann's and Weber-Riemann's *Partielle Differential-Gleichungen*, and to many of Lord Kelvin's writings. In the first few chapters they have also drawn from Preston's *Heat*, while the general arrangement of the material — proceeding from the simpler to the more complex problems, and with the treatment of Fourier's series deferred till such cases as can be handled without it are completed — is borrowed from the chapter on Heat Conduction in Christiansen's *Theoretical Physics*, to which acknowledgment is also due for material. They have also drawn, especially for tables, from the articles by Graetz in Winkelmann's *Handbuch*, while acknowledgments to a number of original sources are scattered through the text.

In conclusion, the authors take pleasure in thanking their many engineering and geological friends who have contributed information and advice, while they are particularly indebted to Professors Max Mason and H. W. March for many useful suggestions.

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we consider another, making an angle ϕ with it, we can see that both the flux across this surface and the temperature gradient along the normal to such surface will be diminished by the factor $\cos\phi$, so that we may write in general for the flux across any surface

$$W = -k\frac{\partial\theta}{\partial n}, \qquad (3)$$

where the derivative is taken along the outward-drawn normal; that is, in the direction of decreasing temperature.

While the rate at which heat is transferred in a body, for example, along a thermally insulated rod, is dependent only on the conductivity and other factors noted, the rise in temperature which this heat will produce will vary with the specific heat c and density ρ of the body. We must then introduce another constant, h^2, whose significance will be considered later, determined by the relation

$$h^2 = \frac{k}{c\rho}. \qquad (4)$$

The constant h^2 has been termed by Kelvin the *diffusivity* of the substance, and by Maxwell its *thermometric conductivity.*

Equations (1) and (3) express what is sometimes referred to as the fundamental hypothesis of heat conduction. Its justification or proof rests on the agreement of calculations made on this hypothesis, with the results of experiment, not only for the very simple but for the most complicated cases as well.

3. Field of Application. From equation (1) we may infer in what field the results of our study will find their application. We may conclude, first, that our derivations will hold good for any body in which heat transference takes place according to this law, if k is the same for all parts and all directions in the body. This includes all homogeneous isotropic solids, and also liquids and gases in cases where convection and radiation are negligible. The equation also shows that, since only *differences* of temperature are involved, the *actual* temperature of the system is immaterial. We shall have frequent cause to remember this statement, for, while many cases are derived on the supposition

that the temperature at the boundary is zero, the results are ma applicable to cases in which this is any other constant temperatu by a simple shift of the temperature scale.

But the results of the study of heat conduction are not limit in their application to heat alone, for parts of the theory fi application in certain gravitational problems, in static and curre electricity, and in elasticity, while the methods developed are very general application in mathematical physics. As an exar ple of such relationship to other fields it may be pointed o that if θ in (1) is interpreted as electrical potential, and k electrical conductivity, we have the law of the flow of electricity and all our derivations may be interpreted accordingly.

4. Units; Dimensions. There is probably no subject in whicl the confusion of units is greater than that of heat conduction While the physicist uses the metric or C.G.S. unit, — that is the (gram) calorie per second, per square centimeter of area, fo a temperature gradient of a degree centigrade per centimeter,— there is no such uniformity of practice among engineers. Th steam engineer refers his observations to the B.T.U.* per hour per square foot, per degree Fahrenheit, per inch in thickness while the refrigerating engineer prefers the day as the unit o time rather than the hour, and the electrical engineer uses variou systems, based frequently on the kilowatt, as representing th rate of heat flow. There are also numbers of other units,† some of them‡ making use of the idea of thermal resistance analogous to electrical resistance, and therefore being reciprocall related to conductivity. These various engineering units hav been introduced to simplify the computation of heat losses ir various types of problems, and on these grounds perhaps justif their existence; but from the standpoint of the present worl

* The British Thermal Unit (B.T.U.) is the quantity of heat required to raise th temperature of one pound of water one degree Fahrenheit, at its temperature o maximum density (39° F.).

† Norton, "Thermal Properties of Concrete," *Proc. Nat. Assoc. of Cement Users* **7**, p. 89 (1911), mentions that he has even seen a report in terms of hogsheads of wate raised to the boiling point, time not mentioned!

‡ Hering, "The Thermal Ohm and the Thermal Mho," *Met. and Chem. Eng.* **9**, p. 13 (1911).

they are, with one or two exceptions, not usable. This is because, in a large majority of the cases we shall have occasion to consider, it is not the conductivity but the *diffusivity*, or thermometric conductivity, which enters directly into the computations, and this latter is too complex a unit (see Art. 2) to use profitably with a mixture of English and metric systems, or an English system involving two different units of length — for example, feet and inches, as in common engineering practice. Only two, then, of the many heat-conduction units lend themselves readily to our purpose — the B.T.U. per hour, per square foot, for a temperature gradient of a degree Fahrenheit per foot, and the metric unit. But the former is practically never used (the gradient being expressed in degrees per *inch* in the common engineering unit), while the latter is becoming of more general use every day, so we shall confine our units and calculations to the metric system, giving in many cases, however, the English equivalents.

The following relations and transference factors may prove of use:

1 m.	= 39.37 in. = 3.2808 ft. = 1.0936 yd.
1 in.	= 2.540 cm.
1 sq. m.	= 10.764 sq. ft. = 1.196 sq. yd.
1 sq. in.	= 6.452 sq. cm.
1 kg.	= 2.2046 lb.
1 lb.	= 453.6 g.
1 B.T.U.	= 252.0 cal.
1 watt	= 0.2389 cal. per second.
1 kilowatt	= 56.88 B.T.U. per minute.
1 watt per square foot	= 3.413 B.T.U. per square foot per hour.
1 cal. per square centimeter	= 3.687 B.T.U. per square foot.
1 cal. per square centimeter per second	= 318,500 B.T.U. per square foot per day.
Temperature in centigrade	= $\frac{5}{9}$ (temperature in Fahrenheit minus 32°).

To reduce conductivity expressed in B.T.U. per hour, per square foot, per degree Fahrenheit, per inch, to metric units, divide by 2903. If the day is the unit of time instead of the

hour, divide by 69,670. If thermal resistivity expressed in "thermal ohms per centimeter cube" * is given, the conductivity in metric units is deduced by multiplying the reciprocal of the resistivity by 0.2389. A diffusivity expressed, as in some English writings, in terms of French feet and year units is reduced to metric by multiplying by 0.00003346. In other cases the transference factor is readily arrived at from a consideration of the dimensions of the units. From (1),

$$k = \frac{Q}{\theta_1 - \theta_2} \cdot \frac{x}{At}. \tag{5}$$

Now since the unit of heat is that necessary to raise unit mass of water one degree, its dimensions are mass and temperature, so the dimensions of $\frac{Q}{\theta_1 - \theta_2}$ are simply $[M]$. Hence

$$[k] = \frac{[M]}{[L][T]}, \tag{6}$$

so that in another system in which the units are M', L', T', the number k' which represents the conductivity in this system is determined by

$$k' = k\frac{M}{M'} \cdot \frac{T'}{T} \cdot \frac{L'}{L}. \tag{7}$$

Similarly, it is easily shown for the diffusivity that

$$h'^2 = h^2 \frac{T'}{T} \cdot \frac{L^2}{L'^2}. \tag{8}$$

In some cases the unit of heat is taken as that which will raise unit volume, rather than unit mass, of water one degree. The above relations then become

$$k' = k\frac{T'}{T} \cdot \frac{L^2}{L'^2}, \tag{9}$$

and

$$h'^2 = h^2 \frac{T'}{T} \cdot \frac{L^5}{L'^5} \cdot \frac{M'}{M}. \tag{10}$$

5. Values of the Constants. In Appendix A is given a table of the conductivity constants for a number of substances, including

* Hering, *Met. and Chem. Eng.*, **9**, p. 652 (1911).

most of the metals. The conductivity and diffusivity are not absolute constants, but depend in some degree on external conditions, chiefly on the temperature, hence this is specified where possible. Most pure metals show a small and nearly linear decrease of conductivity with increase of temperature, although a few show the reverse effect, as also do many alloys. For moderate ranges of temperature the variation, however, is small (for iron, nickel, and copper the decrease is about 2% in going from 18° C. to 100° C.; aluminum shows an increase of about the same amount), but for higher ranges it is by no means negligible. Angell * finds the conductivity of nickel to reduce by a half when the temperature is raised from 0° C. to 700° C., while aluminum, on the other hand, shows a conductivity at 600° C. more than double that at 0° C.

Nonmetallic substances show an even greater temperature effect, and Nusselt † seems to find that for many materials the conductivity is nearly proportional to the absolute temperature, that is, increases by $\frac{1}{273}$ for every degree above 0° C.

When possible, this change of the thermal constants with temperature should be taken account of in calculations, and this may be done approximately by using the conductivity for the average temperature involved. ‡ The diffusivity usually shows a smaller change with temperature than does the conductivity, for in many cases the specific heat increases with the conductivity so as to leave their quotient very nearly the same. This seems to be the case for concrete. §

There is a very noticeable relation between the thermal and electrical conductivities of the metals, and this has given rise to the so-called law of Wiedemann and Franz, which states that the one is proportional to the other. While this holds in a general way where different metals are under consideration, it does not

* *Phys. Rev.*, **33**, p. 430 (1911).

† *Ztschr. d. Ver. deutscher Ing.*, **52**, p. 906 (1908).

‡ For a more rigorous discussion of this point see Hering, "Effects of the Variations of Thermal Resistivity with Temperature," *Trans. Am. Elect. Chem. Soc.*, **21**, p. 511 (1912).

§ See Norton, "Thermal Properties of Concrete," *Proc. Nat. Assoc. Cement Users*, **7**, p. 78 (1911).

express the facts when a single metal at several different temperatures is concerned, for the electrical conductivity decreases with rise of temperature, while the thermal conductivity is more nearly constant. L. Lorenz * took account of this fact and expressed it in the law that the ratio of thermal divided by electrical conductivity increases for any given metal proportionally to the absolute temperature. It holds only for pure metals with any degree of approximation and only for very moderate temperature ranges.

* L. Lorenz, *Wied. Ann.*, **13**, p. 422 (1881).

CHAPTER II

THE FOURIER CONDUCTION EQUATION

6. Differential Equations. In any mathematical study of heat conduction use must continually be made of differential equations, both ordinary and partial. These occur, however, only in a few special forms whose solutions can be explained as they appear, so only a brief general discussion of the subject is necessary here.

Differential equations are those involving differentials or differential coefficients, and are classified as *ordinary* or *partial*, according as the differential coefficients have reference to one, or to more than one, independent variable. A solution of such an equation is a function of the independent variables which satisfies the equation for all values of these variables. For example,

$$y = \sin x + c$$

is a solution of the simple differential equation

$$dy = \cos x dx.$$

The *general solution*, as its name implies, is the most general function of this sort which satisfies the differential equation, and will always contain arbitrary, that is, undetermined, constants or functions. A *particular solution* may be obtained by substituting particular values of the constants or functions in the general solution. But while this is theoretically the method of obtaining the particular solution, we shall find in practice that in many cases where it would be almost impossible to obtain the general solution of the differential equation, we are still able to arrive at the desired result by combining particular solutions which can be obtained directly by various simple expedients.

7. A differential equation is *linear* when it is of the first degree with respect to the dependent variable and its derivatives. It is also *homogeneous* if, in addition, there is no term

which does not involve this variable or one of its derivatives. Practically all the differential equations we shall have occasion to use are both linear and homogeneous, as are indeed a large share of those occurring in all work in mathematical physics. As examples we may mention the following partial differential equations which are both linear and homogeneous:

Laplace's equation, of constant use in the theory of potential,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0;$$

the equation of the vibrating cord,

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2};$$

the Fourier conduction equation,

$$\frac{\partial \theta}{\partial t}=h^2\left\{\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}+\frac{\partial^2 \theta}{\partial z^2}\right\}.$$

8. If such an equation, that is, linear and homogeneous, is written so that all the terms are on the left side, the right-hand member being consequently reduced to zero, a very useful proposition can at once be deduced. For any value of the dependent variable which satisfies the equation must reduce the left-hand term to zero, so if such particular solution is multiplied by a constant, it will still reduce this term to zero, since it will be merely equivalent to multiplying each term by the constant. In the same way it can be seen that the sum of any number of particular solutions will still be a solution. We may then state as a general proposition that, *in the case of the linear, homogeneous differential equation (ordinary or partial), any combination formed by adding particular solutions, with or without multiplication by arbitrary constants, is still a solution.* We shall have frequent occasion to make application of this law.

9. Boundary Conditions. The solution of practically all heat-conduction problems involves the determination of the temperature θ as a function of the time and space coördinates. Such value of θ is assumed to be a finite and continuous function of

x, y, z, and t, and must satisfy not only the general differential equation, which in one modification or another is common to all heat-conduction problems, but also certain equations of condition which are characteristic of each particular problem. Such are:

(a) *Initial Conditions.* These express the temperature throughout the body at the instant which is chosen as the origin of the time coördinate, as a function of the space coördinates, that is,

$$\theta_0 = f(x, y, z) \text{ when } t = 0.$$

(b) *Boundary or Surface Conditions.* These are of several sorts according as they express: (1) the temperature on the boundary surface as a function of time, position, or both, that is,

$$\theta_S = f(x, y, z, t);$$

(2) that at the surface of separation of two media there is continuity of flow of heat, expressed by the relation

$$k_1\left(\frac{\partial \theta_1}{\partial n}\right) = k_2\left(\frac{\partial \theta_2}{\partial n}\right);^*$$

or (3) that radiation takes place at the surface, in which case we have

$$-k\left(\frac{\partial \theta}{\partial n}\right)_S = E\theta,$$

assuming Newton's law of cooling and calling E the emissivity, that is, the rate of loss of heat by radiation and convection per square centimeter of surface, per degree above the temperature of the surroundings.

Newton's law states that the rate of cooling is proportional to the temperature, θ, above the surroundings, and, when this temperature difference is small, can be shown to be consistent with Stefan's law of radiation

$$R = C(T^4 - T_0^4),$$

where T and T_0 are the absolute temperatures of the radiating body and of the surrounding walls respectively. For with small values of $T - T_0$ we have

$$T^4 - T_0^4 = \delta(T^4) = 4\, T_0^3 \cdot \delta T, \text{ or } R = 4\, CT_0^3 \cdot \delta T.$$

* See equation (3), Chapter I.

Noting that δT is equivalent to θ in Newton's law, we see that this law holds with any degree of accuracy for only small values of θ. However, at ordinary atmospheric pressures and for small differences of temperature, the convection part of the loss is much greater than the radiation.

There are also other possible boundary conditions, such as that the bounding surface shall be impermeable to heat, all of which we shall have frequent occasion to use and shall treat more at length when they occur. Following a common practice, we shall hereafter refer to both initial and surface conditions as simply "boundary conditions."

10. Our task in general, then, in solving any given heat-conduction problem is to attempt, by building up particular solutions of the general conduction equation, to secure one which shall satisfy the given boundary conditions. It is easy to see that such a result is *one* solution of our problem, and it may be shown that it is also the *only* solution, but for the proof of this uniqueness theorem the reader is referred to larger treatises on the subject.*

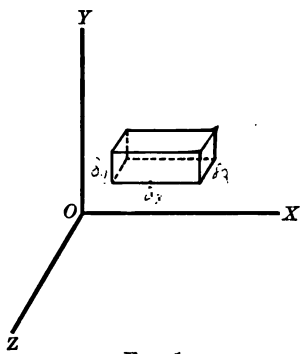


FIG. 1

11. The Fourier Equation. Before taking up the treatment of specific problems we must deduce a general equation of conduction which, with its modifications, will determine the flow of heat under any conditions.

Choose three mutually rectangular axes of reference OX, OY, and OZ (Fig. 1) in any isotropic body and consider a small rectangular parallelepiped of edges δx, δy, and δz parallel respectively to these three axes. Call the temperature at the center of this element of volume θ; then since the temperature will in general be variable throughout the body, we may express its value on any face o the parallelepiped — this being so small that the temperatur is sensibly uniform over any one face — as being greater or le

* Carslaw, *Fourier's Series and Integrals*, p. 206.

than this mean temperature θ by a small amount. The magnitude of this small amount for the case of the $\delta y\delta z$ faces we may readily show to be

$$\frac{1}{2}\frac{\partial\theta}{\partial x}\delta x, \tag{1}$$

for the temperature gradient $\frac{\partial\theta}{\partial x}$ measures the change of temperature per unit length along OX, and the distance of $\delta y\delta z$ from the center is evidently $\frac{1}{2}\,\delta x$. Then the temperature of the right- and left-hand faces may be written

$$\theta+\frac{1}{2}\frac{\partial\theta}{\partial x}\,\delta x \quad\text{and}\quad \theta-\frac{1}{2}\frac{\partial\theta}{\partial x}\,\delta x \text{ respectively.} \tag{2}$$

Now since equation (1) of Chapter I, which defines the coefficient of conductivity, may be written in the differential form as

$$\frac{\partial Q}{\partial t}=-kA\frac{\partial\theta}{\partial x}, \tag{3}$$

then the flow of heat per second in the positive x direction through the left-hand face $\delta y\delta z$ is

$$-k\delta y\delta z\frac{\partial}{\partial x}\left(\theta-\frac{1}{2}\frac{\partial\theta}{\partial x}\,\delta x\right), \tag{4}$$

and through the right-hand face in the same direction

$$-k\delta y\delta z\frac{\partial}{\partial x}\left(\theta+\frac{1}{2}\frac{\partial\theta}{\partial x}\,\delta x\right), \tag{5}$$

the negative sign being used, since a positive flow of heat evidently requires a negative temperature gradient. The difference of these two quantities is evidently the gain in heat of the element due to the x component of flow alone; then since similar expressions hold for the other two pairs of faces, the differences of these three pairs of expressions, or

$$k\frac{\partial^2\theta}{\partial x^2}\,\delta x\delta y\delta z+k\frac{\partial^2\theta}{\partial y^2}\,\delta x\delta y\delta z+k\frac{\partial^2\theta}{\partial z^2}\,\delta x\delta y\delta z, \tag{6}$$

represents the difference between the total inflow and total outflow of heat, or the amount by which the heat of the element is

being increased per second. If the specific heat of the material of the body be c and its density ρ, this sum must equal

$$c\rho\delta x\delta y\delta z\frac{\partial\theta}{\partial t}. \tag{7}$$

Hence we may write

$$k\left\{\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right\}=c\rho\frac{\partial\theta}{\partial t}; \tag{8}$$

or, since $h^2=\dfrac{k}{c\rho}$,

$$\frac{\partial\theta}{\partial t}=h^2\left\{\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right\}, \tag{9}$$

which is frequently written

$$\frac{\partial\theta}{\partial t}=h^2\nabla^2\theta. \tag{10}$$

This is known as Fourier's equation. It expresses the conditions which govern the flow of heat in a body, and the solution of any particular problem in heat conduction must first of all satisfy this equation, either as it stands, or in a modified form.

CHAPTER III

STEADY STATE—ONE DIMENSION

12. A body in which heat is flowing is said to have reached a *steady state* when the temperatures of its different parts do not change with time. Such a state occurs in practice only after the heat has been flowing for a long while. Each part of the body then gives up on one side as much heat as it receives on the other, and the temperature is therefore independent of the time t, although it varies from point to point in the body, being a function of the coördinates x, y, and z. For the steady state, then, Fourier's equation (9), Chapter II, becomes

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}=0. \tag{1}$$

We shall investigate one or two applications of this equation for the case of flow in the x direction only.

13. One-Dimensional Flow of Heat. This includes the common cases of flow of heat through a thin plate or along a rod, the two faces of the plate, or ends of the rod, being at constant temperatures θ_1 and θ_2, and in the latter case the surface of the rod being protected so that heat can enter or leave only at the ends. It also includes the case of the steady flow of heat in any body such that the isothermal surfaces, or surfaces of equal temperature, are parallel planes.

For these cases the general equation of conduction reduces to

$$\frac{d^2\theta}{dx^2}=0, \tag{2}$$

the ordinary derivative being written instead of the partial, since in the case of only a single independent variable a partial derivative would have no particular significance. This integrates into

$$\theta=Ax+B. \tag{3}$$

The constants A and B are determined from the boundary conditions for this case, which are that the temperature is θ_1 at the face of the plate (or end of the bar) whose distance from the yz-plane may be called l, and θ_2 for the face of distance m; or, as these conditions may be simply expressed,

$$\text{(a)}\ \theta = \theta_1 \text{ at } x = l; \quad \text{(b)}\ \theta = \theta_2 \text{ at } x = m. \tag{4}$$

Therefore $\theta_1 = Al + B$ and $\theta_2 = Am + B$. Evaluating A and B, we get as the temperature at any point in the plate distant x from the yz-plane

$$\theta = \frac{(m\theta_1 - l\theta_2)}{(m-l)} - \frac{(\theta_1 - \theta_2)x}{(m-l)}. \tag{5}$$

If we define the rate of flow of heat W as the rate at which heat energy is being transferred through unit cross section, then equation (3), Chapter II, becomes

$$W = -k\frac{\partial \theta}{\partial x}, \tag{6}$$

which in this case gives us

$$W = \frac{k(\theta_1 - \theta_2)}{(m-l)} = \frac{k(\theta_1 - \theta_2)}{u}, \tag{7}$$

where u is the thickness of the plate or the length of the rod.

14. This, which is essentially the simplest case considered in heat conduction, has been made the basis of a number of methods of determining coefficients of conductivity. One of the first was that of Péclet,* who used a plate whose two faces were in contact with streams of flowing water at two different temperatures. The heat conducted through the plate was calculated from the change in temperature of the flowing water, and the fall in temperature taken as the difference of that of the two streams; although, as has been shown since, this is only approximately true and may involve a large error.

15. A more usable method is that of Gray,† who experimented with small rods of the material to be studied. The rod, which was perhaps 6 cm. long and 3 mm. in diameter, was fastened on the one end to a copper hot-water bath and on the other to a

* E. Péclet, *Ann. chim. phys.*, (3), **2**, p. 107 (1841); *Ann. Phys. Chem.*, **55**, p. 167 (1842).
† *J. H. Gray, Proc. Roy. Soc. London*, **56**, p. 199 (1894).

copper sphere of some 5.5 cm. diameter. A thermometer inserted in a small hole in the sphere gave its rise of temperature, from which, knowing its heat capacity, the rate of transfer of heat could be computed and the conductivity determined by the aid of (7).

16. Steady Flow of Heat in a Long Thin Rod. This case differs from the preceding in that losses of heat by radiation and convection are supposed to take place from the sides of the bar and must be taken into account in our calculation. To do this we must add to the Fourier equation $\frac{\partial\theta}{\partial t} = h^2\frac{\partial^2\theta}{\partial x^2}$ a term which shall represent this loss of heat. Now by Newton's law of cooling the rate of this loss will be proportional to the excess of temperature of the surface element over that of the surrounding medium which we shall assume to be at zero, and hence may be represented by $b^2\theta$ where b^2 is a constant. Fourier's equation for this case then becomes

$$\frac{\partial\theta}{\partial t} = h^2\frac{\partial^2\theta}{\partial x^2} - b^2\theta, \tag{8}$$

and when the steady state has been reached, this reduces to

$$\frac{\partial^2\theta}{\partial x^2} = \frac{b^2}{h^2}\theta. \tag{9}$$

This is readily solved by the usual process of substituting e^{ax} for θ, which gives

$$a^2e^{ax} = \frac{b^2}{h^2}e^{ax}, \tag{10}$$

from which we get $a = \pm\frac{b}{h}$, and hence

$$\theta = Ae^{+\frac{b}{h}x} + Be^{-\frac{b}{h}x} \tag{11}$$

as the sum of two particular solutions.

17. The significance of the constant b^2 is most easily shown by considering the problem entirely independently of Fourier's equation. For when the steady state has been reached in such a bar, the flow of heat per unit of time across any area of cross section S of the bar will be, at the point x,

$$-kS\frac{d\theta}{dx}, \tag{12}$$

and, at the point $x + \delta x$,

$$-kS\frac{d}{dx}\left(\theta + \frac{d\theta}{dx}\delta x\right), \tag{13}$$

and consequently the excess of heat left in the bar between these two points δx apart is

$$kS\frac{d^2\theta}{dx^2}\delta x. \tag{14}$$

This must escape by loss from the surface, and such loss per unit of time will be given by $E\theta p\delta x$, where E is the so-called surface emissivity of the bar, that is, its rate of radiation and convection loss per unit of surface per degree of temperature above the surroundings * (see Art. 9), and where $p\delta x$ is the product of the perimeter p of the bar and the length δx of the element, that is, the element of surface. Hence we have

$$kS\frac{d^2\theta}{dx^2} = E\theta p, \tag{15}$$

or

$$\frac{d^2\theta}{dx^2} = \frac{Ep}{kS}\cdot\theta. \tag{16}$$

By comparison with equation (9) we then see that

$$b^2 = \frac{h^2Ep}{kS}. \tag{17}$$

Writing for shortness, say, $\frac{Ep}{kS} = \mu^2$, our general solution (11) takes the form

$$\theta = Ae^{+\mu x} + Be^{-\mu x}. \tag{18}$$

18. We may use this solution to investigate the state of temperature in a long bar, whose far end has the same temperature as the surrounding medium, while the near end is at θ_1, say, the temperature of the furnace. The point at which an intermediate temperature θ_2 is reached must also be known.

Here the boundary conditions are

$$\begin{aligned} &(a)\ \theta = 0 \quad \text{at } x = \infty, \\ &(b)\ \theta = \theta_1 \quad \text{at } x = 0, \\ &(c)\ \theta = \theta_2 \quad \text{at } x = l. \end{aligned} \tag{19}$$

* For values of E, see Appendix A.

From condition (a) we get

$$0 = Ae^{\mu\infty} + Be^{-\mu\infty}, \tag{20}$$

so that

$$Ae^{\infty} = 0 \text{ or } A = 0. \tag{21}$$

Condition (b) then gives

$$\theta_1 = Be^{-\mu 0} \text{ or } B = \theta_1, \tag{22}$$

and (c) means that

$$\theta_2 = \theta_1 e^{-\mu l} \text{ or } \mu l = \log \frac{\theta_1}{\theta_2}. \tag{23}$$

$$\therefore\ \theta = \theta_1 \left(\frac{\theta_1}{\theta_2}\right)^{-\frac{x}{l}}. \tag{24}$$

For different bars subject to the same conditions (a) and (b) and having the same temperature θ_2 at points l_1, l_2, l_3, . . . we have

$$\log \frac{\theta_1}{\theta_2} = \mu_1 l_1 = \mu_2 l_2 = \mu_3 l_3 \cdots = \text{a constant}, \tag{25}$$

which, from the definition of μ, means that

$$\frac{k_1}{l_1^2} = \frac{k_2}{l_2^2} = \frac{k_3}{l_3^2} = \cdots \frac{k_n}{l_n^2}, \tag{26}$$

providing the several bars have each the same perimeter, cross section, and coefficient of emission.

19. This is the fundamental equation underlying the so-called Ingen-Hausz experiment for comparing the conductivities of different metals. The metals, in the form of rods of the same size and character of surface, are coated thinly with beeswax (melting point θ_2) and are placed with one end in a bath of hot oil at temperature θ_1. After standing for some time the wax is found to be melted for a certain definite distance (l) on each bar, and the conductivities are therefore in the ratio of the squares of these distances.

20. Another application of (18) is found in the solution for the case of the bar, heated as above, with the temperatures known at three equidistant points.*

* See Preston, *Heat*, p. 634.

21. These principles were made use of by Despretz * in a series of experiments on the relative conductivities of bars of various metals; also by Wiedemann and Franz,† and, to a certain extent, by Jäger and Diesselhorst,‡ whose determinations of the conductivities of pure metals are probably the best we have. Forbes § sought to measure absolute as well as relative conductivities by this type of experiment, with the aid of a separate test of the rate of cooling of the bar in air.

APPLICATIONS.

22. There could be pointed out an almost unlimited number of practical applications of these deductions for the steady flow of heat in one dimension, particularly of equation (7), but as these are treated at length in general physics and engineering works, and especially in texts on furnaces, boilers, refrigeration, and the like, we shall be content with a few common examples.

23. Ice House. As a first example consider the heat flow through the walls of an ice house. Suppose the walls to be of double $\frac{7}{8}''$ boarding on each side of an $8''$ core of shavings, making with the weatherproof paper a wall $12''$ (30.5 cm.) thick. If the two sides are at temperatures of 0° C. and 21.1° C. (32° F. and 70° F.), what is the rate of heat flow through the wall?

The conductivity for such material, if dry, is about 0.00023‖; hence, from equation (7),

$$W = \frac{.00023 \times 21.1}{30.5} = .000159 \text{ cal. per square centimeter per second}$$
$$= 50.7 \text{ B.T.U. per square foot per day.}$$

The flow per day (86,400 sec.) through such a wall of 1000 sq. m. (1196 sq. yd.) would be sufficient to melt 1720 kg., or 3790 lb., of ice.

* *Ann. chim. phys.*, (2), **19**, p. 97 (1821); **36**, p. 422 (1827).
† *Ann. chim. phys.*, (3), **41**, p. 107 (1854).
‡ *Abh. d. phys.-tech. Reichsanstalt*, **3**, p. 269 (1900).
§ *Phil. Trans. Roy. Soc. Edinburgh*, **23**, p. 133 (1862).
‖ Deduced from Cooper, *Practical Cold Storage*, p. 81.

24. Furnace Walls. What is the loss of heat through a furnace wall 45.7 cm. (18″) thick if the two faces are at 1000° C. and 120° C. (1832° F. and 248° F.), assuming an average conductivity of 0.0024 * for the masonry of the wall?

Here we have

$$W = \frac{.0024 \times 880}{45.7} = .0462 \text{ cal. per square centimeter per second,}$$
$$\text{or 180 watts per square foot.}$$

Despite its small conductivity, air, as in the air space frequently built in such a furnace wall, proves in practice a poor insulator, as pointed out by Ray and Kreisinger.† For at high temperatures the transmission of heat through the air space by radiation is more rapid than if the space were filled with some poorly conducting material such as ashes or sand. For ordinary temperatures, however, as in a refrigerator wall or house wall, such an air space is quite effective, especially if convection is prevented by breaking up the air space into small parts. It is undoubtedly to the entrapped air that porous substances such as cotton, wool, or feathers owe their good heat-insulating properties.

25. In any practical consideration of heat transference it is disastrous to overlook what is generally termed the contact or surface resistance which is offered to the heat flow by any discontinuity in the material. Thus brick masonry, as in a thick wall, shows hardly more than half the conductivity of the brick itself, while powdered brick dust may have ten times the insulating quality of the solid material.

As pointed out by Hering,‡ this consideration is of the highest practical importance in many cases. Thus for a boiler plate of $\frac{1}{4}''$ material (.635 cm.) a flow of 1 kilowatt per square foot (.257 cal. per second per square centimeter) through the plate

* Clement and Elgy (Univ. of Ill. Eng. Exp. Station, *Bull. No. 36*) give .0024 as the conductivity of fire brick at 700° C. Although it is frequently quoted somewhat higher than this (about .0040), the above is probably a fair average for a masonry wall partly of fire brick and partly of ordinary brick.

† "The Flow of Heat through Furnace Walls," *Bull. No. 8*, Bureau of Mines, Department of Interior.

‡ *Met. and Chem. Eng.*, 10, p. 40 (1912).

would necessitate a temperature difference of less than 1.5° C. between the faces, which is inconsiderable compared with the temperature difference between the hot gas and the metal. That this temperature drop through the metal may be of considerable importance, however, is proved by the common experience that a thick-walled hot-air furnace is generally less efficient than one with thinner (for example, sheet-steel) walls.

PROBLEMS

1. Compute the heat loss per day through 100 sq. m. of brick wall ($k = .0020$) 30 cm. thick, if the inner face is at 20° C. and the outer at 0° C. How much coal must be burned to compensate this loss, if the heat of combustion is 7000 cal./gram and the efficiency of the furnace 60%?

(11.5 × 10^7 cal.; 27.4 kg.)

2. In determining the conductivity of iron by Gray's method the following data was obtained: diameter of iron rod, 3 mm.; length, 8 cm.; temperature of hot end, 95.6° C.; mean temperature of copper sphere, 21.2° C.; mass of sphere, 603 g.; rate of rise of temperature, .1008° per minute. What is the conductivity of the iron rod? (0.141.)

3. A long nickel bar is heated at one end. It is found by the aid of inserted thermometers that at a point where the temperature is 65° above that of the room the variation of temperature gradient is .246° per centimeter per centimeter. A separate experiment on a small section of the same bar showed that when this was heated to 65° the rate of cooling was 2.25° per minute. What is the diffusivity of this sample of nickel? (This is the principle of Forbes's method.) (0.152.)

CHAPTER IV

THE STEADY STATE — MORE THAN ONE DIMENSION

26. Flow of Heat in a Plane. We shall solve Fourier's problem of the permanent state of temperatures in a thin rectangular plate of infinite length, whose surfaces are insulated. Call the width of the plate π and suppose that the two long edges are kept constantly at the temperature zero, while the one short edge, or base, is kept at temperature unity. Heat will then flow out from the base to the two sides and toward the infinitely distant end, and our problem will be to find the temperature at any point.

Take the plate as the xy-plane with the base on the x-axis and one side as the y-axis. Then equation (9) of Chapter II becomes

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0. \tag{1}$$

To solve this problem, then, we must find a value for the temperature at any point which shall not only be a solution of (1), but shall also satisfy the boundary conditions for this case, which are

$$\begin{aligned}
&(a)\ \theta=0 \text{ at } x=0,\\
&(b)\ \theta=0 \text{ at } x=\pi,\\
&(c)\ \theta=1 \text{ at } y=0,\\
&(d)\ \theta=0 \text{ at } y=\infty.
\end{aligned} \tag{2}$$

We shall attempt to find a simple particular solution of (1) which will satisfy all the conditions of (2), but, failing this, it may still be possible to combine several particular solutions, as explained in Art. 8, to secure one which will do this.

27. Of the several ways of arriving at such a particular solution we may outline two. The first is with the aid of a device which always succeeds when the equation is linear and homogeneous with constant coefficients. This is to assume that

$$\theta=e^{\alpha y+\beta x}, \tag{3}$$

where α and β are constants. Substituting this in (1), we find at once that

$$\alpha^2 + \beta^2 = 0, \tag{4}$$

which is then the condition to be satisfied in order that $\theta = e^{\alpha y + \beta x}$ may be a solution. But this means that

$$\theta = e^{\alpha y \pm \alpha x i}, \tag{5}$$

for any value of α, is a solution, which is equivalent to saying that

$$\theta = e^{\alpha y} e^{\alpha x i} \tag{6}$$

and

$$\theta = e^{\alpha y} e^{-\alpha x i} \tag{7}$$

are solutions, and by Art. 8 their sum or difference divided by any constant must be a solution also. Then since *

$$e^{i\phi} + e^{-i\phi} = 2\cos\phi, \tag{8}$$

and

$$e^{i\phi} - e^{-i\phi} = 2i\sin\phi, \tag{9}$$

we get, upon adding (6) and (7) and dividing by 2,

$$\theta = e^{\alpha y}\cos\alpha x, \tag{10}$$

and upon subtracting and dividing by $2i$,

$$\theta = e^{\alpha y}\sin\alpha x. \tag{11}$$

Now obviously (10) does not satisfy condition (a) of (2), so we turn to (11), which can be seen at once to satisfy (a) and (b), also (d) if α is negative. As it stands (11) fails to meet condition (c), but it may still be possible to combine a number of particular solutions of the type of (11) which will do this. For if n is any positive integer,

$$\theta = Ae^{-ny}\sin nx \tag{12}$$

fulfills the first, second, and last of the above conditions, as will also the sum

$$\theta = A_1 e^{-y}\sin x + A_2 e^{-2y}\sin 2x + A_3 e^{-3y}\sin 3x + \cdots \tag{13}$$

$$* \quad e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots,$$

while

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots,$$

and

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \cdots.$$

Putting $x = i\phi$, where i is written for $\sqrt{-1}$, we see from these that $e^{i\phi} = \cos\phi + i\sin\phi$, and $e^{-i\phi} = \cos\phi - i\sin\phi$, from which (8) and (9) follow at once.

re A_1, A_2, $\cdots$ are constant coefficients. For $y = 0$ this becomes

$$\theta = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \cdots, \qquad (14)$$

if it is possible to develop unity in such a series, we may be able to satisfy (c) of (2). Now, as we shall discuss at length Chapter VI, Fourier showed that such a development in a onometric series *is* possible, the expression in this case being

$$1 = \frac{4}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right\} \qquad (15)$$

all values of x between 0 and π. Therefore our required ition is

$$\theta = \frac{4}{\pi}\left\{e^{-y}\sin x + \frac{1}{3}e^{-3y}\sin 3x + \frac{1}{5}e^{-5y}\sin 5x + \cdots\right\} \qquad (16)$$

ch satisfies (1) as well as all the boundary conditions of (2).

8. In the second method of solving (1) we shall separate variables at once by assuming that $\theta = X \cdot Y$ where X is a ction of x only, and Y of y only. Substituting, we obtain

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0, \qquad (17)$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -\frac{1}{X}\frac{\partial^2 X}{\partial x^2}. \qquad (18)$$

e the two sides of this equation are functions of entirely in-endent variables, they can be equal only if each is equal to a stant which we may call λ^2. The solution of the partial dif-ntial equation (1) is thus reduced to that of the two ordinary erential equations

$$\frac{d^2Y}{dy^2} - \lambda^2 Y = 0, \qquad (19)$$

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0. \qquad (20)$$

These may be solved by substitutions similar to (3) but some-t simpler, namely,

$$Y = e^{\beta y} \quad \text{and} \quad X = e^{\alpha x} \text{ respectively.} \qquad (21)$$

The first gives $\beta = \pm \lambda$, therefore

$$Y = Ae^{\lambda y} + Be^{-\lambda y}. \tag{22}$$

The second gives $\alpha = \pm i\lambda$, so that

$$X = A'e^{i\lambda x} + B'e^{-i\lambda x}, \tag{23}$$

which, from the note to Art. 27, reduces, if we call $(A' - B')i = D$ and $A' + B' = C$, to

$$X = C \sin \lambda x + D \cos \lambda x. \tag{24}$$

Choosing $A = D = 0$ to satisfy (a), (b), and (d) of (2), the solution resulting from the product of (22) and (24) reduces at once to (12), and the remainder of the process is the same as before.

29. It may be noted that this same sort of solution will hold even if the temperature θ of the base of the plate is other than unity, indeed even if it ceases to be constant and is instead a function of x, provided it can be expressed also in this latter case by a Fourier series. In case we wish to have the values of x run from 0 to a instead of from 0 to π, we must introduce as a variable the quantity $\frac{\pi x}{a}$, and the expressions will otherwise be the same as before. We shall discuss this at length in Chapter VI.

It is also of interest to note that our solution is entirely independent of the physical constants of the medium, so that the temperature at any point is independent of what material is used, so long as the steady state exists.

30. Flow of Heat in a Sphere. To investigate the radial flow of heat in a sphere, we must first replace the rectilinear coördinates x, y, and z in equation (9), Chapter II, by the single variable r. This is done by means of the following transformation:

$$\frac{\partial \theta}{\partial x} = \frac{d\theta}{dr}\frac{\partial r}{\partial x}, \quad \text{and this equals} \quad \frac{d\theta}{dr} \cdot \frac{x}{r}, \tag{2}$$

because, since $\quad r^2 = x^2 + y^2 + z^2, \; \frac{\partial r}{\partial x} = \frac{x}{r};$

also

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{d^2\theta}{dr^2} \cdot \frac{x^2}{r^2} + \frac{d\theta}{dr} \cdot \frac{1}{r} - \frac{d\theta}{dr} \cdot \frac{x^2}{r^3}, \tag{26}$$

with similar expressions for the derivatives with respect to y and z. We thus obtain

$$\nabla^2\theta \equiv \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} = \frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr}. \tag{27}$$

Since, however,

$$\frac{d(r\theta)}{dr} = r\frac{d\theta}{dr} + \theta, \quad \text{and} \quad \frac{d^2(r\theta)}{dr^2} = r\frac{d^2\theta}{dr^2} + 2\frac{d\theta}{dr}, \tag{28}$$

we have

$$\nabla^2\theta = \frac{1}{r}\frac{d^2(r\theta)}{dr^2}. \tag{29}$$

The Fourier equation thus becomes for the steady state

$$\frac{1}{r}\frac{d^2(r\theta)}{dr^2} = 0, \tag{30}$$

and its integral may at once be written as

$$\theta = A + \frac{B}{r}. \tag{31}$$

For boundary conditions we may take

$$\begin{aligned} &\text{(a)}\ \theta = \theta_1 \text{ at } r = r_1, \\ &\text{(b)}\ \theta = \theta_2 \text{ at } r = r_2, \end{aligned} \tag{32}$$

where r_1 and r_2 are respectively the internal and external radii of the hollow sphere. These conditions give, on substitution in (31), after the elimination of A and B,

$$\theta = \frac{r_2\theta_2 - r_1\theta_1}{r_2 - r_1} + \frac{r_1r_2(\theta_1 - \theta_2)}{r(r_2 - r_1)}. \tag{33}$$

This expresses the temperature for any point of the sphere and also shows that the isothermal surfaces are concentric spheres. The rate of flow of heat per unit area in the direction r is given by

$$W = -k\frac{d\theta}{dr} = \frac{k(\theta_1 - \theta_2)\,r_1r_2}{r^2(r_2 - r_1)}, \tag{34}$$

and the total quantity which flows out in unit time is

$$4\pi r^2 W = \frac{4\pi k(\theta_1 - \theta_2)\,r_1r_2}{r_2 - r_1}. \tag{35}$$

31. Radial Flow of Heat in a Cylinder. Let the axis of the cylinder be the z-axis. Then the problem is similar to that for the sphere, save that now we are concerned with only two

dimensions and may put $r^2 = x^2 + y^2$. By a process similar to that by which (27) and (29) were obtained we then get

$$\nabla^2\theta = \frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} = \frac{1}{r}\frac{d\left(r\frac{d\theta}{dr}\right)}{dr} = 0. \tag{36}$$

The integral of this is

$$\theta = A \log r + B, \tag{37}$$

which gives, by the use of boundary conditions quite similar to those of (32),

$$\theta_1 = A \log r_1 + B; \text{ then } \theta_2 = A \log r_2 + B, \tag{38}$$

and from these we obtain

$$\theta = \frac{(\theta_1 - \theta_2)\log r}{\log r_1 - \log r_2} + \frac{\theta_1 \log r_2 - \theta_2 \log r_1}{\log r_2 - \log r_1}. \tag{39}$$

The rate of flow is given by

$$W = \frac{k(\theta_1 - \theta_2)}{r(\log r_2 - \log r_1)}, \tag{40}$$

and the quantity of heat which flows out through unit length of the cylinder per second by

$$2\cancel{\pi r}W = \frac{\cancel{2\pi}k(\theta_1 - \theta_2)}{\log r_2 - \log r_1}. \tag{41}$$

32. Niven* has developed an ingenious method of measuring conductivities based on this case. Heat is supplied by a wire carrying an electrical current along the axis of the cylinder, and when a steady state has been reached, the conductivity is given in terms of the difference of temperature at known distances from the axis. These temperatures may be determined by small thermoelectric junctions embedded in the material at the desired places, while the rate at which the heat is being supplied is known from the current and the difference of potential at the two ends of the wire. Thus for a current I amperes and a potential drop of E volts in length l of the wire the heat lost per centimeter length of the cylinder is as follows:

* *Proc. Roy. Soc.*, 76, p. 34 (1905).

$$\frac{EI}{4.19}\text{ cal./sec.} = \frac{2\pi k(\theta_1-\theta_2)}{\log_e \dfrac{r_2}{r_1}} = \frac{2\pi k(\theta_1-\theta_2)}{2.3026\log_{10}\dfrac{r_2}{r_1}}. \qquad (42)$$

The method is especially useful for the study of cement, rock, and other poor conductors.* The material may also be used in the spherical form and the conductivity determined with the aid of (35).

APPLICATIONS

33. Equation (41) applies at once to the flow of heat through the walls of a boiler flue, although in this case the thickness is so small in comparison with the radius that it can be treated as the flow through a plate (Art. 13, equation (7)). That (40) reduces to this if we put

$$\log r_2 - \log r_1 = d(\log r) = \frac{dr}{r},$$

is evident.

34. Covered Steam Pipes. A better application is found in the case of the covered steam pipe, for the thickness of the covering in this case is frequently comparable with the diameter of the pipe itself. As an application of (41) let us investigate the loss of heat per unit length of a 2″ steam pipe (outside diam. 2.375″, or 6.04 cm.), protected with a special magnesia covering 2.54 cm. (1″) thick. Let the temperature of the pipe be 185° C. (365.2° F.), that is, 150 lb. steam pressure; and of the outer surface of the covering, 47.2°C. (117°F.). The conductivity of the covering will be taken in this case as 0.000156.

Then

$$2\pi r W = \frac{2\pi\times .000156\times 137.8}{2.3026\log_{10}\dfrac{5.56}{3.02}} = .221\text{[illegible]. per second per centimeter length of pipe}$$

$$= 96.2\dagger\text{ B.T.U. per hour per foot length.}$$

* For the description of a method analogous to this but applicable to good conductors, see paper by M. F. Angell, *Phys. Rev.*, **33**, p. 421 (1911). The material is used in the form of a rod which is heated by the passage of a large current. The temperature gradient is obtained from thermocouples at the center and at the surface.

† This is the result obtained in a test by Barrus under the above conditions. See C. P. Paulding, "Loss of Heat from Covered Steam Pipes," Stevens Indicator, 19, p. 110 (1902).

It is of interest to note that double this thickness of covering would still allow a loss of .137 cal./sec. per centimeter length for the same temperature range, or only 38% decrease in loss for 158% added covering material. That the proportional saving is greater for a larger pipe is shown by the curves of Fig. 2.

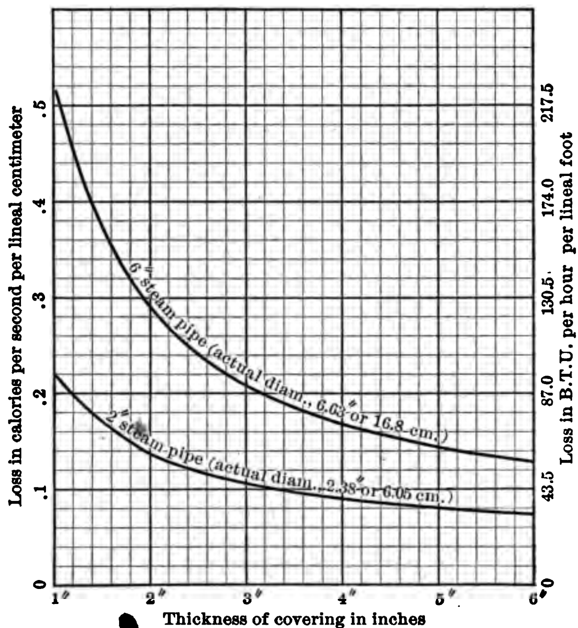


FIG. 2. Curves showing the relation of heat loss to thickness of covering, for two sizes of steam pipe. See Art. 34

35. The temperature of current-carrying wires as affected by the insulation is also a question which might be studied with the aid of the foregoing equations. It can easily be shown that a wire insulated with a covering of not too low thermal conductivity may run cooler, for a given current, than the same wire if bare; the insulation in this case produces, effectively, so much more cooling surface. The analogous case for steam pipes * would occur for a covering of perhaps ten times the conductivity of

* *Paulding, Stevens Indicator*, **19**, p. 126 (1902).

magnesia, which might cause a greater loss of heat than would occur from the bare pipe.

36. Equations (34) and (35) or (40) and (41) may also be used to find the approximate heat loss from a "flaring" conductor, such as the corners or edges of a furnace, these being treated approximately as portions of a sphere or cylinder respectively.*

PROBLEMS

1. Plot the temperatures for a dozen points in a plane such as treated in Chapter IV, and draw roughly the isothermals and lines of flow. Use only the first two terms of equation (16).

2. A wire whose resistance per centimeter length is .1 ohm is embedded along the axis of a cylindrical cement tube of radii .05 cm. and 1.0 cm. An electric current of 5 amp. is found to keep a steady difference of 125° between the inner and outer surfaces. What is the conductivity of the cement, and how much heat must be supplied per centimeter length?

(.0023; .596 cal./sec.)

3. A hollow lead sphere whose inner and outer diameters are 1 cm. and 10 cm. respectively is heated electrically by a 10-ohm coil placed within the cavity. From the conductivity of lead given in Appendix A compute what current will keep the two surfaces at a steady difference of temperature of 5°. Also, at what rate must heat be supplied?

(1.10 amp.; 2.89 cal./sec.)

4. Two steam pipes of 20 cm. diameter, protected with 10 cm. thick coverings of concrete ($k = .0022$) and magnesia ($k = .00017$) respectively, are run underneath the soil. If the outer surfaces are at 30° C. and the pipes themselves are at 160° C., compute the losses in the two cases.

(2.59 and .20 cal. per second per centimeter length.)

* For a lengthy treatment of this subject see Hering, "Heat Conductance through Walls of Furnaces," *Trans. Am. Electrochem. Soc.*, **14**, p. 215 (1908); also paper before Boston meeting, April, 1912.

CHAPTER V

PERIODIC FLOW OF HEAT IN ONE DIMENSION

37. We shall now take up the problem of the flow of heat in one dimension which takes place in a medium when the boundary plane, normal to the direction of flow, undergoes simply-periodic variations in temperature. This problem occupies in a way an intermediate place between those of the steady state already considered and the more general cases which can be treated only after a familiarity has been gained with Fourier's series; for in the former cases the temperature at any point has been constant, while in the latter it is a more or less complicated function of the time, rarely reaching the same value twice at a given point; but in the present case the temperature at each point in the medium varies in a simply-periodic manner with the time, and while the temperature condition is by no means "steady," as we have defined this term, it duplicates itself in each complete period.

The problem derives its interest and importance from its very practical applications. The surface of the earth undergoes daily and annual changes of temperature which are nearly simply-periodic, and it is frequently desirable to know at just what time a maximum or minimum of temperature will be reached at any point below the surface, as well as the actual value of this temperature. Such knowledge would be of value, for example, in determining the necessary depth for water pipes, to avoid danger of freezing, or in giving warning of just when to anticipate such danger after the appearance of a "cold wave," that is, one of those roughly periodic variations of temperature which frequently characterize a winter.

38. Solution. Our fundamental equation for this case is the Fourier conduction equation

$$\frac{\partial \theta}{\partial t} = h^2 \nabla^2 \theta \qquad (1)$$

written in one dimension,

$$\frac{\partial\theta}{\partial t} = h^2 \frac{\partial^2\theta}{\partial x^2}, \tag{2}$$

and the solution must fit the boundary condition

$$\theta = \theta_0 \sin \omega t \text{ at } x = 0. \tag{3}$$

As the equation (2) is linear and homogeneous with constant coefficients we can arrive at a particular solution by the same device used in Art. 27, namely, by the assumption that

$$\theta = Ae^{\alpha t + \beta x}. \tag{4}$$

Substitution in (2) shows that this is a solution, provided only that

$$\alpha = h^2\beta^2; \tag{5}$$

so we have as a solution

$$\theta = Ae^{\alpha t \pm \frac{x}{h} \cdot \sqrt{\alpha}}. \tag{6}$$

If α is replaced by $\pm \gamma i$, this becomes

$$\theta = Ae^{\pm \gamma t i \pm \frac{x}{h}\sqrt{\gamma}\sqrt{\pm i}}. \tag{7}$$

But

$$\sqrt{i} = \pm \tfrac{1}{2}\sqrt{2}(1 + i),* \tag{8}$$

and

$$\sqrt{-i} = \pm \tfrac{1}{2}\sqrt{2}(1 - i); \tag{9}$$

so that (7) becomes

$$\theta = Ae^{\pm \gamma t i \pm \frac{x}{h}\sqrt{\frac{\gamma}{2}}(1 \pm i)}, \tag{10}$$

or

$$\theta = Ae^{\pm \frac{x}{h}\sqrt{\frac{\gamma}{2}}} \cdot e^{\pm i\left(\gamma t \pm \frac{x}{h}\sqrt{\frac{\gamma}{2}}\right)}. \tag{11}$$

From the several solutions contained in (11) other particular solutions may be built up by addition, such as

$$\theta = Ae^{-\frac{x}{h}\sqrt{\frac{\gamma}{2}}}\left[e^{i\left(\gamma t - \frac{x}{h}\sqrt{\frac{\gamma}{2}}\right)} - e^{-i\left(\gamma t - \frac{x}{h}\sqrt{\frac{\gamma}{2}}\right)}\right], \tag{12}$$

and from Art. 27 this may be written

$$\theta = Be^{-\frac{x}{h}\sqrt{\frac{\gamma}{2}}} \sin\left(\gamma t - \frac{x}{h}\sqrt{\frac{\gamma}{2}}\right). \tag{13}$$

Other solutions may be formed in the same way, care being taken to note, however, that, from the manner of its formation

* $(1 + i)^2 = 1 + 2i - 1 = 2i. \quad \therefore \sqrt{i} = \pm \tfrac{1}{2}\sqrt{2}(1 + i).$

(see (6)), the sign before i in each term of (10) must be the same. This will be found equivalent to saying that the same sign must be used before $\frac{x}{h}\sqrt{\frac{\gamma}{2}}$ in each term of equations like (12). With this in mind we may write at once as other particular solutions

$$\theta = B'e^{\frac{x}{h}\sqrt{\frac{\gamma}{2}}} \cdot \sin\left(\gamma t + \frac{x}{h}\sqrt{\frac{\gamma}{2}}\right), \tag{14}$$

$$\theta = Ce^{-\frac{x}{h}\sqrt{\frac{\gamma}{2}}} \cdot \cos\left(\gamma t - \frac{x}{h}\sqrt{\frac{\gamma}{2}}\right), \tag{15}$$

and

$$\theta = C'e^{\frac{x}{h}\sqrt{\frac{\gamma}{2}}} \cdot \cos\left(\gamma t + \frac{x}{h}\sqrt{\frac{\gamma}{2}}\right). \tag{16}$$

Of these four solutions, (14) and (16) demand that the temperature increase indefinitely as x increases, which is evidently absurd, while (15) is excluded by (3). Equation (13) will satisfy this condition if B is put equal to θ_0 and γ to ω. Making these changes, we have then as the solution

$$\theta = \theta_0 e^{-\frac{x}{h}\sqrt{\frac{\omega}{2}}} \cdot \sin\left(\omega t - \frac{x}{h}\sqrt{\frac{\omega}{2}}\right), \tag{17}$$

which expresses the temperature for any time t at any distance x from the surface.

39. Amplitude, Range. This is the equation of a wave motion whose rapidly decreasing amplitude is given by the factor $\theta_0 e^{-\frac{x}{h}\sqrt{\frac{\omega}{2}}}$. The range of temperature, or maximum variation, for any point below the surface is given by

$$R = 2\,\theta_0 e^{-\frac{x}{h}\sqrt{\frac{\omega}{2}}} = 2\,\theta_0 e^{-\frac{x}{h}\sqrt{\frac{\pi}{T}}}, \tag{18}$$

putting for ω its value $\frac{2\pi}{T}$, where T is the period. θ_0 is the amplitude, or half range, at the surface. This shows at once that the slower the variation of temperature the greater the range in the interior of the body.

40. Lag, Velocity, Wave Length. The time at which a maximum or minimum of temperature will occur at any point is evidently that for which

$$\omega t - \frac{x}{h}\sqrt{\frac{\omega}{2}} = (2n+1)\frac{\pi}{2}, \tag{19}$$

or

$$t = \frac{\frac{x}{h}\sqrt{\frac{\omega}{2}} + (2n+1)\frac{\pi}{2}}{\omega}, \tag{20}$$

odd values of n giving minima, and even, maxima. Fixing our attention on the minimum which occurs at the surface when, say, $\omega t = \frac{3\pi}{2}$, we see that if x and t are both supposed to increase so that

$$\omega t - \frac{x}{h}\sqrt{\frac{\omega}{2}} = \frac{3\pi}{2}, \tag{21}$$

we may think of this particular minimum being propagated into the medium and reaching any point x at the time given by this equation. This is later than its occurrence at the surface by an amount

$$t = \frac{x}{h}\sqrt{\frac{1}{2\omega}} = \frac{x}{2h}\sqrt{\frac{T}{\pi}}, \tag{22}$$

which may be called the *lag* of the temperature wave. The same reasoning holds for the maximum, or zero, or any other phase.

The apparent velocity of such a wave in the medium is given from (22) by

$$V = \frac{x}{t} = 2h\sqrt{\frac{\pi}{T}}, \tag{23}$$

but this is merely the rate at which a given maximum or minimum may be said to travel, and has nothing to do with the actual speed with which the heat energy is transmitted from particle to particle, for this latter is very large — possibly the same as the velocity of transmission of sound in the substance.

From (23) we may deduce as the expression for the wave length of such a wave

$$\lambda = VT = 2h\sqrt{\pi T}. \tag{24}$$

Equations (22)–(24) may be used to measure the diffusivity of any medium from determinations of the lag, velocity, or wave length.

41. Temperature Curve in the Medium. The form of this curve at any given time may be conveniently investigated by differentiating (17) with respect to x to find the maxima and minima of the curve, which, of course, will be distinguished from the maxima and minima above treated. Then writing

$$\mu = \frac{1}{h}\sqrt{\frac{\omega}{2}},$$

we have
$$\tan(\omega t - \mu x) = -1, \tag{25}$$

or
$$x = \frac{\left(\frac{\pi}{4} + \omega t\right)}{\mu}, \frac{\left(\frac{5\pi}{4} + \omega t\right)}{\mu}, \frac{\left(\frac{9\pi}{4} + \omega t\right)}{\mu}, \dots. \tag{26}$$

This shows that the minima and maxima are equally spaced, and if we note that the corresponding minima and maxima of the pure sine curve
$$y = \sin(\omega t - \mu x) \tag{27}$$

occur at
$$x = \frac{\left(\frac{\pi}{2} + \omega t\right)}{\mu}, \frac{\left(\frac{3\pi}{2} + \omega t\right)}{\mu}, \dots, \tag{28}$$

they are seen to be nearer the surface than these latter by an amount $\frac{\pi}{4\mu}$. This means that when $t = nT$ (or $(n + \frac{1}{2})T$) the first minimum (or maximum) is found at just half the distance of the corresponding minimum (or maximum) for the sine curve. This is illustrated in the solid line curve in Fig. 3, which gives the temperatures for different depths for the diurnal wave in soil of diffusivity $= .0049$. The broken line is the curve of amplitudes for an amplitude, or half range, of 5° at the surface.

42. Flow of Heat per Cycle through the Surface. This is readily computed by forming the temperature gradient from (17) and then integrating over the half period. Thus

$$\frac{\partial\theta}{\partial x} = \theta_0 e^{-\frac{x}{h}\sqrt{\frac{\omega}{2}}} \cdot \left(-\frac{1}{h}\sqrt{\frac{\omega}{2}}\right) \cdot \left\{\sin\left(\omega t - \frac{x}{h}\sqrt{\frac{\omega}{2}}\right) + \cos\left(\omega t - \frac{x}{h}\sqrt{\frac{\omega}{2}}\right)\right\}, \tag{29}$$

and $$H = -k\int_0^{\frac{T}{2}}\left(\frac{\partial\theta}{\partial x}\right)_{x=0} dt = -k\int_0^{\frac{\pi}{\omega}}\left(\frac{\partial\theta}{\partial x}\right)_{x=0} dt$$

$$= \theta_0 \frac{k}{h}\sqrt{\frac{T}{\pi}} \text{ cal. per square centimeter.} \tag{30}$$

This amount of heat flows through the surface into the material during one half the cycle and out again during the other half.

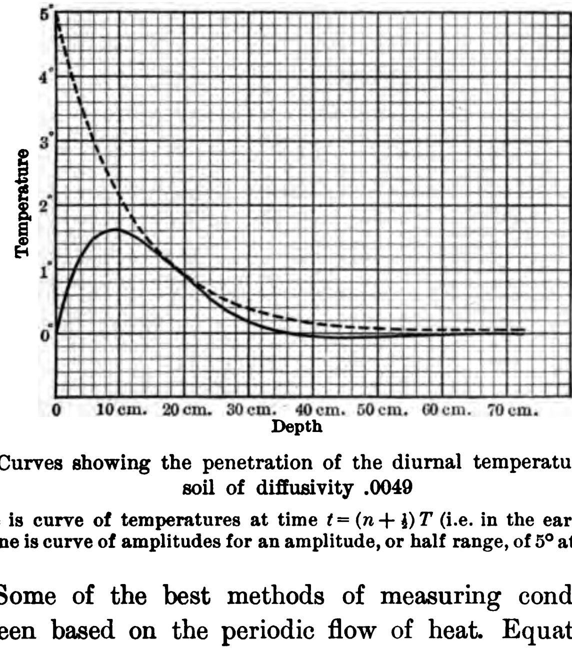


FIG. 3. Curves showing the penetration of the diurnal temperature wave in soil of diffusivity .0049

Solid line is curve of temperatures at time $t = (n + \frac{1}{4})T$ (i.e. in the early evening). Broken line is curve of amplitudes for an amplitude, or half range, of 5° at the surface

43. Some of the best methods of measuring conductivities have been based on the periodic flow of heat. Equation (18) shows that the diffusivity h^2 can be determined if the period and range of temperature are measured at a distance x, the range at the surface being known. Or this last condition can be eliminated if the range is known for two or more distances. This method was utilized by Forbes,* who measured the annual variations of temperature for different depths of soil and rock near Edinburgh, and thus determined their thermal constants. Ångström † has

* Forbes, *Trans. Roy. Soc. Edinburgh*, **16**, pt. II (1846). See also Kelvin, *Math. and Phys. Papers*, III, p. 261.

† Ångström, *Phil. Mag.*, **25**, p. 130 (1863); and **26**, p. 161. See also *Pogg. Ann.*, **114**, p. 513 (1861); and *Ann. chim. phys.*, (3), **67**, p. 379.

made extensive use of the periodic method in determining the conductivity of metals in the form of bars. These had thermometers inserted in holes at frequent intervals along their lengths and were heated and cooled in the middle. As radiation took place from the surfaces of the rods, and as, moreover, the temperature was not a simply-periodic function of the time, the preceding equations had to be modified to take account of these facts.*

APPLICATIONS

44. With the aid of the foregoing equations we may investigate the penetration of periodic temperature waves into the earth. The questions of interest and importance in this connection are, first, the range or variation of temperature at various depths for the diurnal and annual changes; and, second, the velocity of penetration of such waves, and hence the time at which the maximum or minimum may be expected to occur at various depths.

45. Diurnal Wave. First consider the diurnal wave. Suppose the surface of the soil to vary daily, at a certain season, from $+16°$ C. to $-4°$ C. (60.8° F. to 24.8° F.), what is the range at 30 cm. (11.8″) and 1 m. (39.4″)? The mean of the above temperatures is $+6°$, and as condition (3) supposes a mean temperature of zero, our temperature scale must be reduced by the subtraction of 6°, which can be added again later if necessary. In this case, then, θ_0 is 10° and $T = 86{,}400$ sec. Using the constants for ordinary moist soil ($h^2 = .0049$), equation (18) shows that the range is reduced from 20° at the surface to only .07 of this, or 1.4° C. (2.5° F.), at 30 cm. below, and to less than .004° C. at 1 m. below. Since a range of 12° would just be sufficient in this case — assuming an average temperature of 6° in the soil — to reach a freezing temperature, we conclude that a layer of soil 6 cm. thick will be enough to prevent freezing under these conditions. Dry soil will afford even smaller penetration than this, and in the damp soil we have neglected the latent heat of freezing of

* See Preston, *Heat*, p. 654; Byerly, *Fourier's Series*, p. 91.

the soil, which, while nearly negligible for small water content, would still reduce the penetration of the freezing temperature somewhat. We may also deduce from (22) that the maximum or minimum temperature at 30 cm. would lag some 35,000 sec., or 9.7 hr., behind that at the surface. In a series of soil temperature measurements by MacDougal * the lag of the maximum at 30 cm. depth was found to be from 8 to 12 hr., and the range generally less than a tenth of the range in air, both figures being in substantial agreement with the above deductions.

46. Annual Wave. For the annual wave the variation for temperate latitudes may be taken as 22° C. to − 8° C. (71.6° F. to 17.6° F.). The range at 1 m. will then be reduced to 19° C., while at 10 m. below the surface it will be only .33° C. The freezing temperature will penetrate to a depth of less than 170 cm. (5.6 ft.).† From (23) the velocity of penetration of such a wave is .000045 cm. per sec., or 3.9 cm. per day. For a soil of this diffusivity, then, the minimum temperature at a depth of about 7 m. (23 ft.) would occur in July and the maximum in January.

The following table is compiled from measurements of underground temperatures in Japan, cited by Tamura.‡ The computed temperature range and lag were calculated for a diffusivity of $h^2 = .0027$ by (18) and (22).

TABLE

Depth	Observed annual range	Calculated annual range	Observed lag	Calculated lag
0.0 cm.	28.2° C.	28.2° C.	0 days	0 days
30 cm.	22.7° C.	23.4° C.	2.5 days	10.6 days
60 cm.	18.7° C.	19.5° C.	9.0 days	21.6 days
120 cm.	14.0° C.	13.5° C.	35.0 days	42.3 days
300 cm.	5.2° C.	4.6° C.	93.5 days	106.0 days
500 cm.	1.3° C.	1.3° C.	177.5 days	176.5 days
700 cm.	.4° C.	.4° C.	267.0 days	247.0 days

* *Monthly Weather Review*, **31**, p. 375 (1903).

† In reality, considerably less than this, because of the latent heat of freezing.

‡ *Monthly Weather Review*, **33**, p. 296 (July, 1905).

It will be noted that for the greater depths, where the soil is doubtless more nearly uniform, the agreement of observed and computed values is good.

47. "Cold Waves." While the preceding formulas were developed on the assumption of a simply-periodic temperature wave which continues indefinitely, they are still applicable with a fair degree of approximation to temporary variations of a roughly periodic nature, such as "cold waves." A good example of this is furnished by observations on underground temperatures by Rambaut.* The curve of temperatures for March, 1899, shows a marked drop, or cold wave, of about 10 days' duration, — whole period 20 days, — the lowering (θ_0) amounting to about 8.6° C. The magnitude of the temperature fall and lag of the minimum, as observed by platinum thermometers at various depths, is given in the table below, and also the computed values. These latter were obtained by using the value of $h^2 = .0074$ computed by Rambaut from the annual-wave curve.

TABLE

Depth	Observed temperature fall	Computed temperature fall	Observed lag	Computed lag
0.0 cm.	8.6° C.	8.6° C.	0 days	0 days
16.5 cm.	5.9° C.	6.7° C.	1.4 days	.8 days
45.7 cm.	3.4° C.	4.2° C.	2.5 days	2.3 days
107.9 cm.	1.3° C.	1.6° C.	4.9 days	5.4 days
174.0 cm.	.33° C.	.57° C.	8.0 days	8.7 days

The computed temperature fall is of course half the range as determined from (18).

48. Temperature Waves in Concrete. The above discussions apply at once to masses of concrete if the diffusivity is taken as .0058 (that is, somewhat more than for average soil, but less than Rambaut found for the soil at Oxford). For example, we may conclude that a cold wave of 3 days' duration (period 6 days), of minimum temperature − 20° C. (− 4° F.), would cause the

* "*Underground* Temperatures at Oxford," *Phil. Trans. Roy. Soc.*, *A*, **195**, p. 235 (*1901*).

freezing temperature to penetrate into a concrete mass or wall at 4° C. (39.2° F.), a depth of 56 cm. (22″), while the yearly variation of temperature at a depth of 2 m. (6.6 ft.) in a mass of concrete (for example, a dam) is only .43 of what it is at the surface.*

49. Periodic Flow in Cylinder Walls. As another instance of the periodic flow may be mentioned the heat penetration in the walls of a steam-engine cylinder. Callendar and Nicolson† found that for 100 revolutions per minute the temperature range of the inner surface of the cylinder wall (cover) during a cycle was 2.8° C. (5.1° F.). Putting for cast iron $h^2 = .121$, we find from (18) that this variation is reduced at a depth of .25 cm. (.1″) to

$$2.8\, e^{-\frac{.25}{\sqrt{.121}}\sqrt{\frac{100\pi}{60}}} = 2.8 \times 10^{-.71} = .54° \text{ C. } (1° \text{ F.}),$$

and at three times this depth to only 0.021° C. (0.04° F.). The heat flow into and out of the walls which takes place each cycle is given from (30) as

$$1.4 \cdot \frac{.108}{.348}\sqrt{\frac{60}{100\,\pi}} = .190 \text{ cal. per square centimeter} \\ = .701 \text{ B.T.U.‡ per square foot.}$$

This discussion of the heat flow into cylinder walls has a direct bearing on the question of steam jacketing such cylinders; for the inflow is attended by a wasteful condensation of steam — followed by evaporation during the subsequent outflow — and the steam jacket is designed to keep the cylinder hot enough to prevent this condensation. There are greatly differing views, however, as to the efficiency of such an arrangement.§

50. Temperature Stresses. As an example of these we may outline briefly what might be termed the problem of the "bonded" street-railway rail. From the preceding calculations it is evident

* For a series of temperature measurements in a concrete bridge see a forthcoming Bulletin of the Iowa Agricultural College Experiment Station. The authors are indebted to Mr. W. D. Maxwell of the city engineer's office, Des Moines, Iowa, for a preliminary report on this work.

† *Proc. Inst. Civ. Eng.*, **131**, p. 147 (1898).

‡ For reasons which are not evident, Callendar and Nicolson use a formula leading to values larger than this by the factor $\sqrt{2}$.

§ See, for example, Kent, *Mech. Eng. Pocketbook*, p. 975 (1910).

that the customary burying of the body of the rail will afford small protection from the annual wave, although it may protect it from the severity of short cold waves and diurnal changes. Assuming that the top of the rail is flush with the surface, and that its temperature at every point is that of the adjoining material (soil, granite blocks, etc.), the average temperature throughout the rail at any time will be given by

$$\Theta = \frac{1}{l}\int_0^l \theta dx,$$

where θ has the value given in (17) and l is the depth of the rail. The maximum or minimum temperature may be found by differentiating this with respect to time and equating to zero. This gives the time at which a maximum or minimum occurs, and its value can be found accordingly. The tension which exists in the rail will be proportional to the difference between its average temperature and that at which it was bonded, and may be arrived at by the following simple reasoning: If a free rail be allowed to cool, it will contract by an amount α, per degree per unit of length, α being the linear coefficient of expansion. To stretch this rail again to its original length would require a tension, per unit area of section, of $P = M\alpha$ units of force for each degree it had cooled, M being Young's modulus. This, then, may be regarded as the tension which will just keep the rail from contraction as it is cooled, and, from the minimum temperature attained, the maximum stress can be readily computed.

It should be realized that this will give at best only a roughly approximate solution of the practical case, for the assumption that the rail takes the temperature of the surrounding materials is questionable, while, of course, it is not the same thickness top and bottom, as tacitly assumed in the above integral. The method might also be used to gain some idea of the temperature stresses set up in the outer layers of a concrete surface, for example, sidewalk; but a more satisfactory way, especially if the concrete be reënforced, is to take these stresses as largely

determined by the temperature gradient at any point. Differentiating (17), we can put this in the form

$$\frac{\partial \theta}{\partial x} = -\frac{\theta_0}{h}\sqrt{\frac{\pi}{T}}\cdot e^{-\frac{x}{h}\sqrt{\frac{\pi}{T}}}\left\{\sin\left(\omega t - \frac{x}{h}\sqrt{\frac{\pi}{T}}\right) + \cos\left(\omega t - \frac{x}{h}\sqrt{\frac{\pi}{T}}\right)\right\}, \quad (31)$$

which shows that temperature stresses due to periodic variations are greatest for the surface layers of the material.

PROBLEMS

1. If the diurnal change of temperature at the surface of a soil of diffusivity .0049 is 20° C., what is the range at 10 cm. and 1 m. below the surface? (8.4°; .0036°.)

2. Solve the preceding problem for an annual range of 30° C., and for points at depths of 10 cm., 1 m., and 10 m. (28.7°; 19.1°; .33°.)

3. Compute the periodic heat flow into and out of the surface for the two preceding problems. (87.6; 2510 cal./cm².)

4. A long copper rod is carefully insulated throughout its length, and one end is alternately heated and cooled through the range 0°–100° every half hour. Plot the temperatures along the bar for such time as will make the heated end at 50°. Determine the wave length and velocity for this case; also for the case in which the period is one-fourth hour.
($\lambda = 160$ cm., $V = .089$ cm./sec. for $T = \frac{1}{2}$ hr.; $\lambda = 113$ cm., $V = .126$ cm./sec. for $T = \frac{1}{4}$ hr.)

5. A cold wave of two weeks' duration ($T = 4$ weeks) brings a temperature fall (amplitude) at the surface of 20°. What will be the fall at a depth of 1 m. in very dry soil ($h^2 = .0031$), in ordinary moist soil ($h^2 = .0049$), and in concrete ($h^2 = .0058$)? Also compute the time lag of the minimum in these cases. (2.5°, 3.9°, 4.4°; 9.1, 7.3, 6.7 days.)

CHAPTER VI

FOURIER'S SERIES

51. Before we can proceed farther with our study of heat-conduction problems we shall be obliged to take up the development of functions in trigonometric series. The necessity for this was apparent in Chapter IV, and could indeed be foreseen in the last chapter; for it was evident that if the boundary condition had been expressed by other than a simple sine or cosine function, as it was, it could not have been satisfied by any of the solutions obtained, unless it should be of such a nature that it could be developed as a series of sine or cosine terms, in which case it might be possible to build up particular solutions to fit it.

Such a development was shown by Fourier to be possible for all functions which fulfill certain simple conditions. For example, the curve $y = f(x)$ may be represented between the limits $x = 0$ and $x = \pi$, by adding a series of sine curves, thus:

$$f(x) = y = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots, \qquad (1)$$

or by a similar cosine series. The $f(x)$ can be represented in this way if it is *single-valued* within the region considered, — that is, if for every x there is one and only one value of y, — and is moreover *finite*, with a finite number of maxima and minima, and *continuous*, or at least has only *finite discontinuities*. The function which represents the initial state of temperature in a body will satisfy these conditions, for there can be but a single value of the temperature at each point of a body, and this value must be finite. Furthermore, while there may exist initial discontinuities, as at a surface of separation between two bodies, such discontinuities will always be finite. This indicates the applicability as well as importance of Fourier's *series* in the theory of heat conduction.

52. Development in Sine Series. To accomplish this development it is necessary to find the values of the coefficients $a_1, a_2, a_3, \cdots$ of the series (1). It is possible to find the value of a finite number, n, of these by solving n equations of the type

$$y_p = a_1 \sin x_p + a_2 \sin 2x_p + \cdots + a_n \sin nx_p, \tag{2}$$

where x_p is one of n particular values of x chosen between 0 and π. This process also has the merit of making plausible the *possibility* of expanding a function in such a series; for with n terms the curve made up by summing the trigonometrical series coincides with the curve $y = f(x)$ at the n points, and can be made identical with it if we take n large enough. But while this method is possible, it is not the simplest way, for the results may be obtained by a much shorter procedure, as follows:

We shall proceed on the assumption that the expansion (1) is possible, and consider this assumption justified if we can find values for the coefficients. Multiply both sides of (1) by $\sin mxdx$, where m is the number of the coefficient we wish to determine; then integrate from 0 to π:*

$$\int_0^\pi f(x) \sin mxdx = a_1 \int_0^\pi \sin mx \sin xdx + \cdots + a_m \int_0^\pi \sin^2 mxdx$$
$$+ \cdots + a_p \int_0^\pi \sin mx \sin pxdx + \cdots. \tag{3}$$

Now $\displaystyle\int_0^\pi \sin mx \sin pxdx$

$$= \frac{1}{2}\int_0^\pi \cos(p-m)xdx - \frac{1}{2}\int_0^\pi \cos(p+m)xdx$$
$$= \frac{1}{2}\left[\frac{1}{p-m}\sin(p-m)x\right]_0^\pi - \frac{1}{2}\left[\frac{1}{p+m}\sin(p+m)x\right]_0^\pi = 0; \tag{4}$$

hence the only term remaining on the right-hand side of (3) is

$$a_m \int_0^\pi \sin^2 mxdx = a_m \frac{\pi}{2}. \tag{5}$$

Therefore $$a_m = \frac{2}{\pi}\int_0^\pi f(x) \sin mxdx, \tag{6}$$

* It can be shown that this procedure is essentially the same as that employed above if n is large. See Byerly, *Fourier's Series*, p. 38.

and the complete series may be written

$$f(x)=\frac{2}{\pi}\left\{\left(\int_0^\pi f(x)\sin xdx\right)\sin x+\left(\int_0^\pi f(x)\sin 2\,xdx\right)\sin 2\,x\right.$$
$$\left.+\cdots+\left(\int_0^\pi f(x)\sin nxdx\right)\sin nx+\cdots\right\}. \quad (7$$

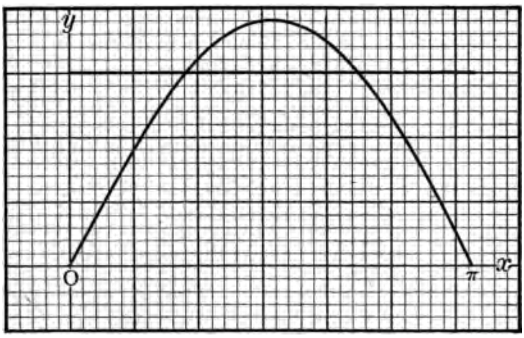


FIG. 4 *a*. One term

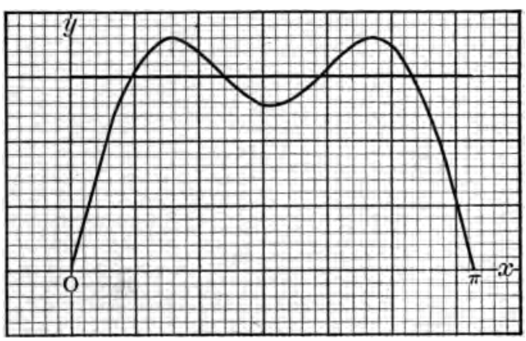


FIG. 4 *b*. Two terms

53. As examples of the application of this series let us develop a few simple functions in this way.

(a) $f(x)=c$, any constant.

$$a_m=\frac{2}{\pi}\int_0^\pi c\sin mxdx=\frac{2\,c}{\pi}\int_0^\pi \sin mxdx \quad (8$$

$$=\frac{2\,c}{\pi m}\Big[1-(-1)^m\Big] \quad (9$$

$$= 0 \text{ if } m \text{ is even} \tag{10}$$

$$= \frac{4c}{\pi m} \text{ if } m \text{ is odd.} \tag{11}$$

ence the even terms will be lacking, and we get

$$f(x) \equiv c = \frac{4c}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]. \tag{12}$$

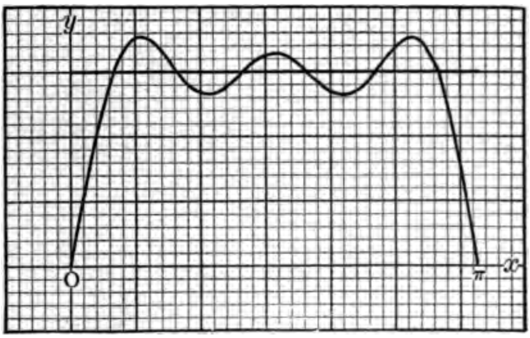


FIG. 4c. Three terms

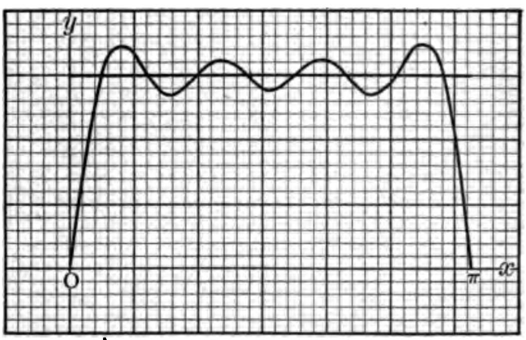


FIG. 4d. Four terms

e approximation curves for the sine series for $y = f(x)$, where $f(x)$ = a constant, $(0 < x < \pi)$

For $x = \frac{\pi}{2}$, this enables us to write the expansion for $\frac{\pi}{4}$ thus:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

(b) Let us reproduce the curve

$$f(x) = x, \text{ from } x = 0 \text{ to } x = \frac{\pi}{2}, \tag{13}$$

$$f(x) = \pi - x, \text{ from } x = \frac{\pi}{2} \text{ to } x = \pi. \tag{14}$$

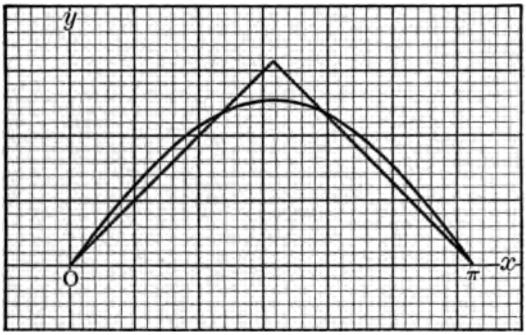


FIG. 5*a*. One term

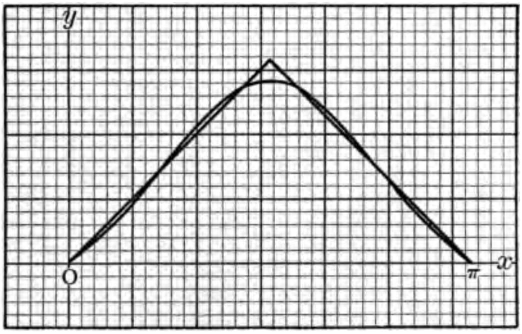


FIG. 5*b*. Two terms

The approximation curves for the sine series for $y = f(x)$, where

$$\left.\begin{aligned} f(x) &= x, \left(0 < x < \frac{\pi}{2}\right) \\ f(x) &= \pi - x, \left(\frac{\pi}{2} < x < \pi\right) \end{aligned}\right\}$$

$$a_m = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x) \sin mx dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x) \sin mx dx \tag{15}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x \sin mx dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin mx dx \tag{16}$$

$$=\frac{2}{\pi}\left\{\left[\frac{\sin mx}{m^2}-\frac{x\cos mx}{m}\right]_0^{\frac{\pi}{2}}+\pi\left[-\frac{1}{m}\cos mx\right]_{\frac{\pi}{2}}^{\pi}\right.$$
$$\left.-\left[\frac{\sin mx}{m^2}-\frac{x\cos mx}{m}\right]_{\frac{\pi}{2}}^{\pi}\right\} \qquad (17)$$
$$=\frac{4}{m^2\pi}\sin m\frac{\pi}{2}. \qquad (18)$$

$$\left.\begin{aligned}
&\text{If } m=1, \text{ or } 4p+1, && \sin m\frac{\pi}{2}=\ \ 1,\\
&m=2, \text{ or } 4p+2, && \sin m\frac{\pi}{2}=\ \ 0,\\
&m=3, \text{ or } 4p+3, && \sin m\frac{\pi}{2}=-1,\\
&m=4, \text{ or } 4p+4, \text{ or } 4p, && \sin m\frac{\pi}{2}=\ \ 0, \text{ where } p \text{ is any integer.}
\end{aligned}\right\} \qquad (19)$$

Again, the even terms are absent, and

$$f(x)=\frac{4}{\pi}\left\{\frac{\sin x}{1}-\frac{\sin 3x}{9}+\frac{\sin 5x}{25}-\cdots\right\}. \qquad (20)$$

(c) Finite discontinuity. Let

$$f(x)=x \text{ from } x=0 \text{ to } x=\frac{\pi}{2} \qquad (21)$$
$$=0 \text{ from } x=\frac{\pi}{2} \text{ to } x=\pi. \qquad (22)$$

Breaking up a_m into two parts and substituting the values for $f(x)$, we get

$$a_m=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}x\sin mx\,dx+\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}0\cdot\sin mx\,dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}x\sin mx\,dx, \qquad (23)$$
$$=\frac{2}{\pi}\left[\frac{\sin mx}{m^2}-\frac{x\cos mx}{m}\right]_0^{\frac{\pi}{2}}, \qquad (24)$$
$$=\frac{2}{\pi}\left[\frac{1}{m^2}\right] \text{ if } m=4p+1,$$
$$=\frac{2}{\pi}\left[\frac{m\pi}{2m^2}\right] \text{ if } m=4p+2,\ =\frac{2}{\pi}\left[-\frac{1}{m^2}\right] \text{ if } m=4p+3, \qquad (25)$$
$$=\frac{2}{\pi}\left[-\frac{m\pi}{2m^2}\right] \text{ if } m=4p+4.$$

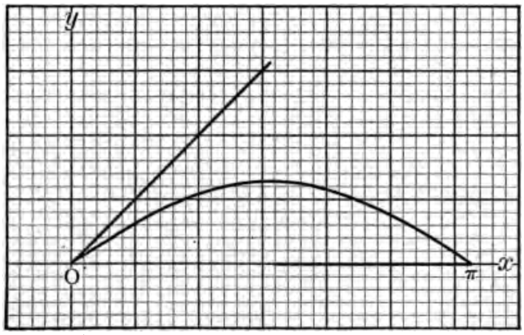


FIG. 6*a*. One term

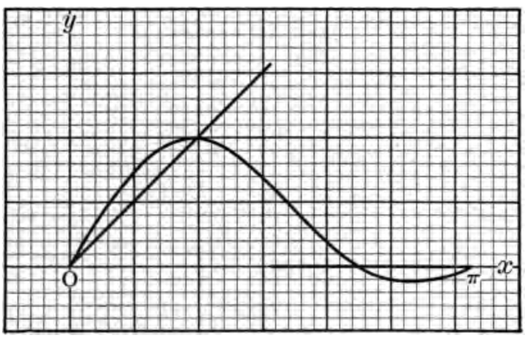


FIG. 6*b*. Two terms

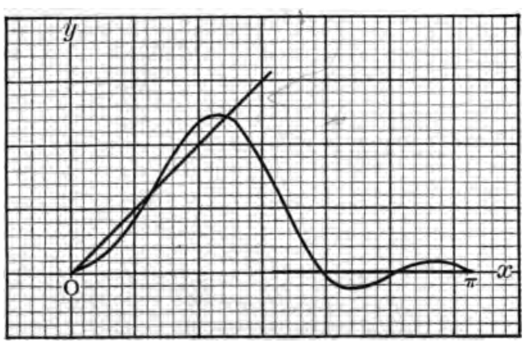


FIG. 6*c*. Four terms

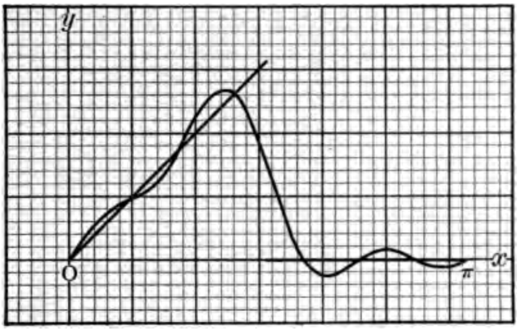


FIG. 6*d*. Six terms

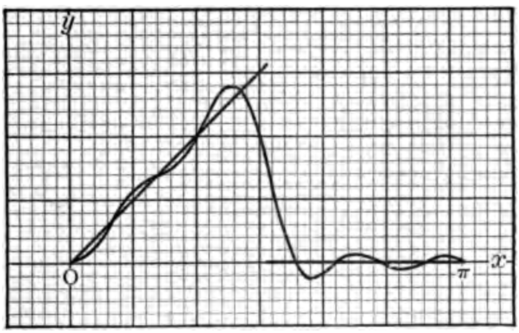


FIG. 6*e*. Eight terms

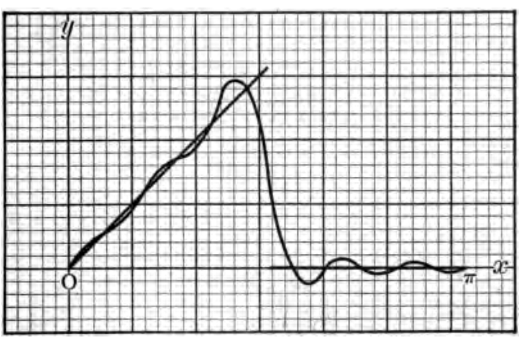


FIG. 6*f*. Ten terms

The approximation curves for the sine series for $y = f(x)$, where

$$\left.\begin{aligned} f(x) &= x, \left(0 < x < \frac{\pi}{2}\right) \\ f(x) &= 0, \left(\frac{\pi}{2} < x < \pi\right) \end{aligned}\right\}$$

$$\therefore f(x) = \frac{2}{\pi}\left\{\frac{\sin x}{1} + \frac{\pi \sin 2x}{4} - \frac{\sin 3x}{9} - \frac{2\pi \sin 4x}{16} + \frac{\sin 5x}{25} + \frac{3\pi \sin 6x}{36} - \cdots\right\}. \quad (26)$$

It may be noted that at the point of discontinuity, $x = \frac{\pi}{2}$, the value of the series is

$$\frac{2}{\pi}\left[\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\right] = \frac{2}{\pi}\left[\frac{\pi^2}{8}\right] = \frac{\pi}{4} \text{ (see Appendix F)}, \quad (27)$$

which is the mean of the values approached by the function as x approaches $\frac{\pi}{2}$ from opposite sides.

54. Development in Cosine Series. In a manner quite similar to the foregoing we are also able to develop such functions as fulfill the conditions we have mentioned, in cosine series between the limits $x = 0$ and $x = \pi$. Thus

$$f(x) = b_0' + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots. \quad (28)$$

The constant term which appears here, though not in the sine series, may be thought of as the coefficient of a term $b_0' \cos(0 \cdot x)$, which shows at once why the corresponding term for the sine series is lacking.

To find the value of any coefficient b_m, we proceed as before, multiplying both sides of (28) by $\cos mxdx$ and integrating from 0 to π; then, since terms of the type

$$\int_0^\pi b_n \cos nx \cos mxdx \quad (29)$$

vanish just as did similar terms in (4), we have remaining on the right-hand side only

$$b_m \int_0^\pi \cos^2 mxdx = \frac{b_m}{2m}\left[(mx + \cos mx \sin mx)\right]_0^\pi \quad (30)$$

$$= \frac{\pi}{2} b_m \text{ if } m \neq 0.$$

$$\therefore b_m = \frac{2}{\pi}\int_0^\pi f(x) \cos mxdx. \quad (31)$$

To get b_0' we must multiply (28) by dx only and integrate from 0 to π; then

$$\int_0^\pi f(x)\,dx = \int_0^\pi b_0' dx + \int_0^\pi b_1 \cos x dx + \cdots = b_0'\pi, \qquad (32)$$

since all terms but the first vanish. Therefore

$$b_0' = \frac{1}{\pi}\int_0^\pi f(x)\,dx. \qquad (33)$$

This is just half the value that (31) would give if $m = 0$ were substituted; therefore, to save an extra formula, (28) is generally written

$$f(x) = \tfrac{1}{2} b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots, \qquad (34)$$

where the value of any coefficient, including the first, is given by (31). The complete cosine series may then be written

$$\begin{aligned} f(x) = \frac{2}{\pi}\bigg\{\frac{1}{2}\int_0^\pi f(x)\,dx &+ \left(\int_0^\pi f(x)\cos x dx\right)\cos x \\ &+ \left(\int_0^\pi f(x)\cos 2x dx\right)\cos 2x + \cdots \\ &+ \left(\int_0^\pi f(x)\cos mx dx\right)\cos mx + \cdots\bigg\}. \qquad (35) \end{aligned}$$

55. As an example take the same function as we developed in a sine series under (b),

$$f(x) = x, \text{ from } x = 0 \text{ to } x = \frac{\pi}{2},$$

$$f(x) = \pi - x, \text{ from } x = \frac{\pi}{2} \text{ to } x = \pi.$$

Then $$b_m = \frac{2}{\pi}\left\{\int_0^{\frac{\pi}{2}} x \cos mx dx + \int_{\frac{\pi}{2}}^\pi (\pi - x)\cos mx dx\right\} \qquad (36)$$

$$\begin{aligned} &= \frac{2}{\pi}\left[\frac{\cos mx + mx \sin mx}{m^2}\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\cdot\frac{\pi}{m}\Big[\sin mx\Big]_{\frac{\pi}{2}}^\pi \\ &\quad - \frac{2}{\pi}\left[\frac{\cos mx + mx \sin mx}{m^2}\right]_{\frac{\pi}{2}}^\pi \text{when } m \neq 0 \qquad (37) \end{aligned}$$

$$=\frac{2}{\pi}\left\{\frac{\cos\frac{m\pi}{2}}{m^2}+\frac{\pi}{2m}\sin\frac{m\pi}{2}-\frac{1}{m^2}-\frac{\pi}{m}\sin\frac{m\pi}{2}\right.$$

$$\left.-\frac{\cos m\pi}{m^2}+\frac{\cos\frac{m\pi}{2}}{m^2}+\frac{\pi}{2m}\sin\frac{m\pi}{2}\right\} \tag{38}$$

$$=\frac{2}{\pi m^2}\left\{2\cos\frac{m\pi}{2}-\cos m\pi-1\right\}. \tag{39}$$

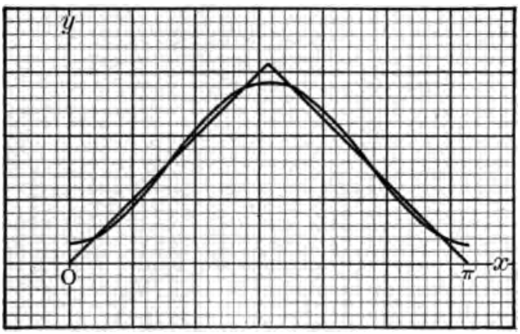


FIG. 7 *a*. Constant term and one other term

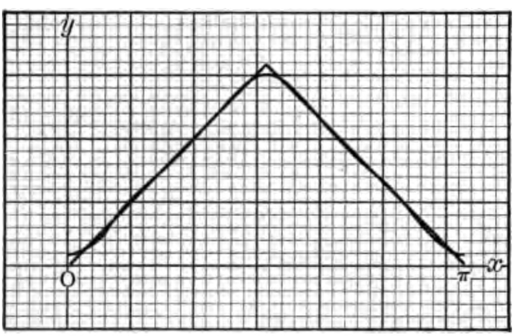


FIG. 7 *b*. Constant term and two other terms

The approximation curves for the cosine series for $y=f(x)$, where

$$\left.\begin{aligned} f(x)&=x, \quad \left(0<x<\frac{\pi}{2}\right)\\ f(x)&=\pi-x, \quad \left(\frac{\pi}{2}<x<\pi\right)\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\text{If } m = 1 \text{ or } 4p+1, \text{ bracket} = 0. \ \therefore\ b_m = 0.\\
&m = 2 \text{ or } 4p+2, \text{ bracket} = -4. \ \therefore\ b_m = -\frac{2}{\pi}\left[\frac{1}{\left(\frac{m}{2}\right)^2}\right].\\
&m = 3 \text{ or } 4p+3, \text{ bracket} = 0. \ \therefore\ b_m = 0.\\
&m = 4 \text{ or } 4p+4, \text{ bracket} = 0. \ \therefore\ b_m = 0.
\end{aligned}\right\} \quad (40)$$

To get b_0, substitute $m = 0$ in (36) and integrate. Then

$$b_0 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\,dx + \frac{2}{\pi}\left[\pi\int_{\frac{\pi}{2}}^{\pi} dx - \int_{\frac{\pi}{2}}^{\pi} x\,dx\right] \quad (41)$$

$$= \frac{\pi}{4} + \pi - \frac{3\pi^2}{4\pi} = \frac{\pi}{2}. \quad (42)$$

$$\therefore\ \frac{1}{2}b_0 = \frac{\pi}{4}. \quad (43)$$

So, finally, we have

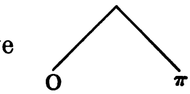
$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right\} \quad (44)$$

to represent the same curve as is given by the sine series (20).

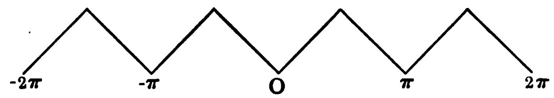
56. Extension of the Limits. We have seen that any function of the kind considered can be represented by either a sine or cosine development for all values of x between 0 and π. We may now question what such series would give at and beyond these limits. Obviously the sine series can hold at the limits $x = 0$ and $x = \pi$ only when the $f(x)$ is itself zero at these points, although it will hold for points infinitesimally near these limits for any value of $f(x)$. For example, it breaks down at the limits in the case of $f(x) = c$ already given.

Both series are periodic and afford curves which must repeat themselves whenever x is changed by 2π, and as both series give the same curve between 0 and π, the difference, if any, between the curves given by the two series must come between π and 2π, or, what amounts to the same thing, between 0 and $-\pi$. This difference is at once evident if we consider that the values

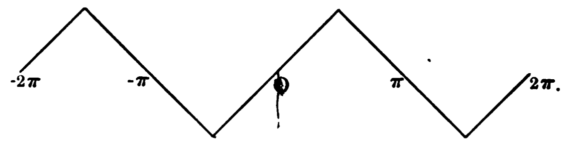
of the sine terms will change sign with change to negative angle, while the cosine terms will not. Thus the cosine development for the curve



gives a periodic curve of the sort



while the sine series gives



57. We may conclude from this, then, that if $f(x)$ is an *even* function, that is, if $f(x) = f(-x)$, it may be represented by a cosine series from $-\pi$ to $+\pi$. Similarly, an *odd* function $(f(x) = -f(-x))$ will be given by a sine series for these same limits. Not all functions are either odd or even, for example, e^x, but it is possible to express any function as the sum of an odd and an even function; thus

$$f(x) \equiv \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}, \tag{45}$$

the first term being even, since it does not change sign with x, while the second does, and is therefore odd. To expand any function satisfying our primitive conditions, then, between $x = -\pi$ and $x = +\pi$, we may write

$$\begin{aligned} f(x) = \tfrac{1}{2} b_0 + b_1 \cos x + b_2 \cos 2x + \cdots \\ + a_1 \sin x + a_2 \sin 2x + \cdots, \end{aligned} \tag{46}$$

where the coefficients are determined by

$$a_m = \frac{2}{\pi} \int_0^\pi \frac{f(x) - f(-x)}{2} \sin mx dx \tag{47}$$

and

$$b_m = \frac{2}{\pi} \int_0^\pi \frac{f(x) + f(-x)}{2} \cos mx dx. \tag{48}$$

Since the values of definite integrals are functions only of the *limits* and not of the variable of integration, we may replace x in

these expressions by any other variable λ; thus

$$a_m = \frac{2}{\pi}\int_0^{\pi} \frac{f(\lambda) - f(-\lambda)}{2} \sin m\lambda d\lambda \tag{49}$$

and

$$b_m = \frac{2}{\pi}\int_0^{\pi} \frac{f(\lambda) + f(-\lambda)}{2} \cos m\lambda d\lambda. \tag{50}$$

It must be kept in mind that the symbol λ, which will be in constant use in the following pages, has merely the significance of a variable of integration, and that we may at any time substitute for it any other variable of integration, for example, α, γ, etc., without in any way affecting the value of the definite integral. Its use must be distinguished from that of x, particularly when we return to the study of conduction problems, for here x usually refers to a particular point or plane in the body.

We can simplify expressions (49) and (50) somewhat, for the former is equivalent to

$$\frac{1}{\pi}\left\{\int_0^{\pi} f(\lambda) \sin m\lambda d\lambda - \int_0^{\pi} f(-\lambda) \sin m\lambda d\lambda\right\}; \tag{51}$$

and if we replace λ by $-\lambda'$ in the second integral, it is transformed into

$$-\int_0^{-\pi} f(\lambda') \sin m\lambda' d\lambda'. \tag{52}$$

This is equal to

$$+\int_{-\pi}^{0} f(\lambda') \sin m\lambda' d\lambda', \tag{52 a}$$

which, from the above discussion of the integration variable, may as well be written

$$+\int_{-\pi}^{0} f(\lambda) \sin m\lambda d\lambda. \tag{53}$$

Hence we have

$$a_m = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(\lambda) \sin m\lambda d\lambda. \tag{54}$$

In a similar way we obtain as the value for b_m

$$b_m = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(\lambda) \cos m\lambda d\lambda. \tag{55}$$

The complete expression (46) is generally known as the true Fourier's series.

58. Change of the Limits. While our expansion as heretofore considered holds only for the region $x = -\pi$ to $x = +\pi$, we can, by a simple change of variable, make it hold from $x = -c$ to $+c$. For let $z = \frac{\pi x}{c}$; then $f(x) = f\left(\frac{cz}{\pi}\right) = F(z)$.

$$\therefore f(x) = F(z) = \tfrac{1}{2} b_0 + b_1 \cos z + b_2 \cos 2z + \cdots$$
$$+ a_1 \sin z + a_2 \sin 2z + \cdots \tag{56}$$

for values of z from $-\pi$ to $+\pi$, and

$$f(x) = \frac{1}{2} b_0 + b_1 \cos \frac{\pi x}{c} + b_2 \cos \frac{2\pi x}{c} + \cdots$$
$$+ a_1 \sin \frac{\pi x}{c} + a_2 \sin \frac{2\pi x}{c} + \cdots \tag{56 a}$$

for values of x from $-c$ to $+c$, where

$$b_m = \frac{1}{\pi} \int_{-\pi}^{+\pi} F(z) \cos mz dz = \frac{1}{c} \int_{-c}^{+c} f(x) \cos \frac{m\pi x}{c} dx, \tag{57}$$

since $z = \frac{\pi x}{c}$, and $dz = \frac{\pi dx}{c}$. This may also be written

$$b_m = \frac{1}{c} \int_{-c}^{+c} f(\lambda) \cos \frac{m\pi\lambda}{c} d\lambda. \tag{58}$$

Similarly,
$$a_m = \frac{1}{c} \int_{-c}^{+c} f(\lambda) \sin \frac{m\pi\lambda}{c} d\lambda. \tag{59}$$

In the same way the sine series (1) may be written

$$f(x) = a_1 \sin \frac{\pi x}{c} + a_2 \sin \frac{2\pi x}{c} + \cdots, \tag{60}$$

where
$$a_m = \frac{2}{c} \int_0^c f(\lambda) \sin \frac{m\pi\lambda}{c} d\lambda; \tag{61}$$

while (34) becomes

$$f(x) = \frac{1}{2} b_0 + b_1 \cos \frac{\pi x}{c} + b_2 \cos \frac{2\pi x}{c} + \cdots, \tag{62}$$

where
$$b_m = \frac{2}{c} \int_0^c f(\lambda) \cos \frac{m\pi\lambda}{c} d\lambda. \tag{63}$$

While series (56 a) applies generally, (60) and (62) hold only from $x = 0$ to c, unless $f(x)$ is an even function, in which case the cosine series will be good from $-c$ to $+c$, while if odd, the sine series will hold over this range.

59. Fourier's Integral. In the foregoing we have developed $f(x)$ into a Fourier's series which represented the function from $-c$ to $+c$ where c may have any value whatever. We shall now proceed to express the sum of such a series in the form of an integral, and, by allowing the limits to extend indefinitely, obtain an expression which holds for all values of x. Write the series (56 a) with the aid of (58) and (59).

$$f(x)=\frac{1}{c}\left\{\frac{1}{2}\int_{-c}^{+c}f(\lambda)\,d\lambda+\int_{-c}^{+c}f(\lambda)\cos\frac{\pi\lambda}{c}\cos\frac{\pi x}{c}\,d\lambda\right.$$
$$+\int_{-c}^{+c}f(\lambda)\cos\frac{2\pi\lambda}{c}\cos\frac{2\pi x}{c}\,d\lambda+\cdots$$
$$+\int_{-c}^{+c}f(\lambda)\sin\frac{\pi\lambda}{c}\sin\frac{\pi x}{c}\,d\lambda$$
$$\left.+\int_{-c}^{+c}f(\lambda)\sin\frac{2\pi\lambda}{c}\sin\frac{2\pi x}{c}\,d\lambda+\cdots\right\}. \tag{64}$$

Collecting terms, this becomes

$$f(x)=\frac{1}{c}\int_{-c}^{+c}f(\lambda)\,d\lambda\left\{\frac{1}{2}+\sum_{m=1}^{m=\infty}\cos\frac{m\pi\lambda}{c}\cos\frac{m\pi x}{c}\right.$$
$$\left.+\sum_{m=1}^{m=\infty}\sin\frac{m\pi\lambda}{c}\sin\frac{m\pi x}{c}\right\}. \tag{65}$$

But since $\cos r\cos s+\sin r\sin s=\cos(r-s)$, this may be written

$$f(x)=\frac{1}{c}\int_{-c}^{+c}f(\lambda)\,d\lambda\left\{\frac{1}{2}+\sum_{m=1}^{m=\infty}\cos\frac{m\pi}{c}(\lambda-x)\right\}; \tag{66}$$

or, if we remember that $\cos(\phi)=\cos(-\phi)$,

$$f(x)=\frac{1}{2c}\int_{-c}^{+c}f(\lambda)\,d\lambda\left\{1+\sum_{m=1}^{m=\infty}\cos\frac{m\pi}{c}(\lambda-x)\right.$$
$$\left.+\sum_{m=-1}^{m=-\infty}\cos\frac{m\pi}{c}(\lambda-x)\right\} \tag{67}$$
$$=\frac{1}{2\pi}\int_{-c}^{+c}f(\lambda)\,d\lambda\left[\frac{\pi}{c}\left\{\sum_{m=-\infty}^{m=+\infty}\cos\frac{m\pi}{c}(\lambda-x)\right\}\right], \tag{68}$$

since $\cos\frac{0\cdot\pi}{c}(\lambda-x)=1$. As c increases indefinitely, we may

write $\gamma = \frac{m\pi}{c}$ and $d\gamma = \frac{\pi}{c}$, and the bracket of (68) then becomes

$$\int_{-\infty}^{+\infty} \cos \gamma (\lambda - x)\, d\gamma. \tag{69}$$

Therefore $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\lambda)\, d\lambda \int_{-\infty}^{+\infty} \cos \gamma (\lambda - x)\, d\gamma, \tag{70}$$

an expression holding for all values of x, and for the same class of functions as previously defined. It is known as Fourier's integral.

60. Equation (70) can be given a slightly different form by means of the following deduction, which will prove of use. For any function, $\phi(\lambda)$,

$$\int_{-c}^{+c} \phi(\lambda)\, d\lambda = \int_{0}^{c} \phi(\lambda)\, d\lambda + \int_{-c}^{0} \phi(\lambda)\, d\lambda. \tag{71}$$

In the last term substitute $-\lambda'$ for λ; then

$$\int_{-c}^{0} \phi(\lambda)\, d\lambda = -\int_{c}^{0} \phi(-\lambda')\, d\lambda' \tag{72}$$

$$= -\int_{c}^{0} \phi(-\lambda)\, d\lambda, \tag{73}$$

since its value is independent of the integration variable (see (53)). If $\phi(\lambda)$ is *even*, that is, if $\phi(\lambda) = \phi(-\lambda)$, (73) means that

$$\int_{-c}^{0} \phi(\lambda)\, d\lambda = -\int_{c}^{0} \phi(\lambda)\, d\lambda = \int_{0}^{c} \phi(\lambda)\, d\lambda, \tag{74}$$

so that $$\int_{-c}^{+c} \phi(\lambda)\, d\lambda = 2\int_{0}^{c} \phi(\lambda)\, d\lambda, \tag{75}$$

while if $\phi(\lambda)$ is *odd*,

$$\int_{-c}^{+c} \phi(\lambda)\, d\lambda = \int_{0}^{c} \phi(\lambda)\, d\lambda - \int_{0}^{c} \phi(\lambda)\, d\lambda = 0. \tag{76}$$

As the cosine is an even function, we may write at once, instead of (70),

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(\lambda)\, d\lambda \int_{0}^{\infty} \cos \gamma (\lambda - x)\, d\gamma. \tag{77}$$

61. Again, if $f(x)$ is either *odd* or *even*, we may put (77) in somewhat simpler form. Since the limits of integration in (77) do not contain either λ or γ, the integration may be performed in either of two possible orders; that is,

$$\int_{-\infty}^{+\infty} f(\lambda)\,d\lambda \int_0^{\infty} \cos\gamma(\lambda - x)\,d\gamma$$

$$= \int_0^{\infty} d\gamma \int_{-\infty}^{+\infty} f(\lambda)\cos\gamma(\lambda - x)\,d\lambda. \qquad (78)$$

Now $\int_{-\infty}^{+\infty} f(\lambda)\cos\gamma(\lambda - x)\,d\lambda$

$$= \int_0^{\infty} f(\lambda)\cos\gamma(\lambda - x)\,d\lambda$$

$$+ \int_{-\infty}^{0} f(\lambda)\cos\gamma(\lambda - x)\,d\lambda; \qquad (79)$$

and following the general methods of the previous article we may write the last term

$$\int_{-\infty}^{0} f(\lambda)\cos\gamma(\lambda - x)\,d\lambda$$

$$= -\int_{\infty}^{0} f(-\lambda')\cos\gamma(-\lambda' - x)\,d\lambda' \qquad (80)$$

$$= \int_0^{\infty} f(-\lambda')\cos\gamma(\lambda' + x)\,d\lambda' \qquad (81)$$

$$= \int_0^{\infty} f(-\lambda)\cos\gamma(\lambda + x)\,d\lambda \qquad (82)$$

$$= -\int_0^{\infty} f(\lambda)\cos\gamma(\lambda + x)\,d\lambda, \text{ if } f(\lambda) \text{ is } \textit{odd}, \qquad (83)$$

$$= \int_0^{\infty} f(\lambda)\cos\gamma(\lambda + x)\,d\lambda, \text{ if } f(\lambda) \text{ is } \textit{even}. \qquad (84)$$

Therefore, if $f(x)$ is *odd*, (77) becomes, for all values of x,

$$f(x) = \frac{1}{\pi}\int_0^{\infty} d\gamma \int_0^{\infty} f(\lambda)\left\{\cos\gamma(\lambda - x) - \cos\gamma(\lambda + x)\right\}d\lambda \quad (85)$$

$$= \frac{2}{\pi}\int_0^{\infty} d\lambda \int_0^{\infty} f(\lambda)\sin\gamma\lambda\,\sin\gamma x\,d\gamma, \qquad (86)$$

while if it is *even*, we have, instead,

$$f(x)=\frac{1}{\pi}\int_0^\infty d\gamma\int_0^\infty f(\lambda)\left\{\cos\gamma(\lambda-x)+\cos\gamma(\lambda+x)\right\}d\lambda \quad (87)$$

$$=\frac{2}{\pi}\int_0^\infty d\lambda\int_0^\infty f(\lambda)\cos\gamma\lambda\cos\gamma x d\gamma. \quad (88)$$

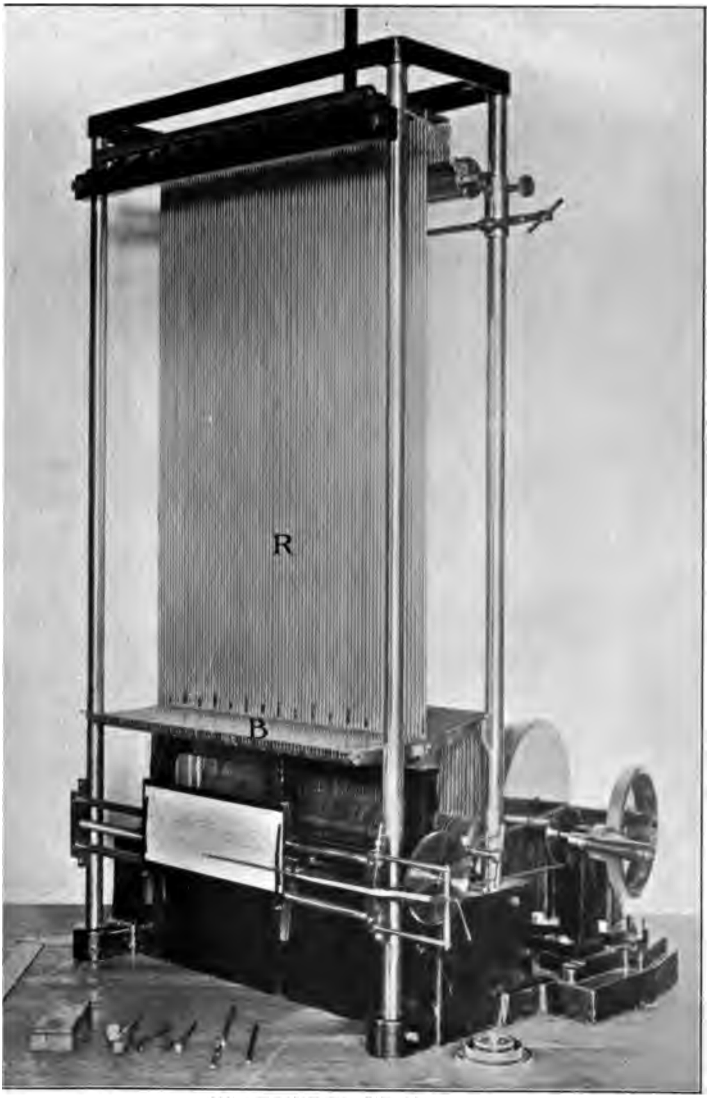
Equations (86) and (88) hold for *all positive* values of x in the case of any function.

62. Harmonic Analyzers. The analytical development of a function in a Fourier's series, with the determination of a large number of coefficients, is well-nigh impossible in many cases, and in any event involves considerable computation. To eliminate this there have been invented several machines which are designed to compound automatically a limited number of sine or cosine terms into the resulting curve, or to perform the more difficult inverse process of analyzing a given function into its component Fourier's series. One of the earliest of these has become well known because of its great simplicity, as well as from the fame of its designer, Lord Kelvin. A long cord is passed over a series of fixed and movable pulleys, to each of which latter a simple harmonic motion of appropriate period and amplitude is given. The end of the cord will then have a displacement at each instant equal to double the sum of the displacements of the movable pulleys. The drawback to such a machine is that the number of elements it is possible to use to advantage is small, because of the unavoidable stretch and elasticity of the cord.

These defects are done away with in a form of analyzer designed by Michelson and Stratton,* in which the component motions are added not by the pulling of a cord but by the stretching of a number of small spiral springs. The machine† illustrated in Fig. 8 has 80 elements, one of which is represented in section in Fig. 9. The eccentric A is arranged by means of gearing to rotate with a period varying from 1 to 80, according

* *Phil. Mag.*, **45**, p. 85 (1898).

† A number of analyzers of this type have been made by Wm. Gaertner & Co., of Chicago.



Courtesy of Professor Michelson and The University of Chicago Press

FIG. 8. The Michelson and Stratton Harmonic Analyzer

to the location of the element in the machine. By means of the rod R, which rests on the lever B, there is produced in the small spring s a harmonic stretching which is proportional in amplitude to d. The 80 small springs s are connected to the pivoted cylinder C, and their additive effect rotates it in varying amounts against the pull of the single stiff spring S'. This motion is amplified by a lever and transmitted by a cord to the pen which writes vertically on paper carried on a plate which is moved at the same time horizontally, so that the result is a curve.

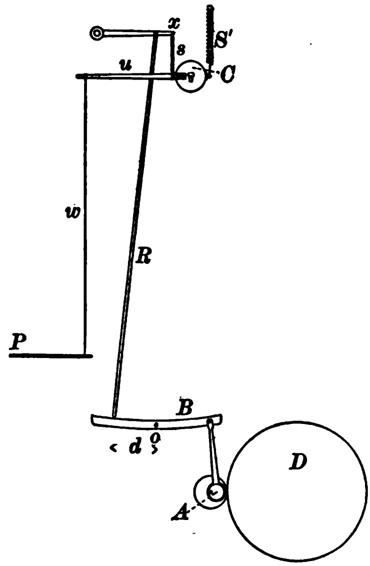


FIG. 9

The process of combining sine or cosine terms is that of adjusting each rod R on its lever B so that it has the proper amplitude d for its period, amplitudes being reckoned positive or negative from the center o. On turning the crank which actuates the machine and at the same time slides the plate holding the paper, the curve represented by the series will be obtained.

63. The method of reversing this process and finding for any given function the coefficients of the corresponding Fourier's series may be seen from the following considerations:

Suppose we wish to develop a function in terms of the sine series. Then

$$f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots, \tag{89}$$

where
$$a_p = \frac{2}{\pi}\int_0^\pi f(x) \sin px dx. \tag{90}$$

Now call the total width of the m elements of the machine π, and the width of the space devoted to each element α, so that

the x-coördinate of the element whose number is n is $n\alpha$. Then if the distance d, or amplitude factor, of each element is made proportional to $f(n\alpha)$, the machine will give the sum

$$\sum_{n=0}^{n=m} f(n\alpha)\sin n\gamma, \tag{91}$$

where γ has any value between 0 and π. But, since n evidently has the value $\frac{x}{\pi}m$, the above may be written

$$\sum_{x=0}^{x=\pi} f(x)\sin\frac{m}{\pi}\gamma x. \tag{92}$$

If $\gamma = \frac{p\pi}{m}$ so that $p = \frac{m\gamma}{\pi}$, the above expression is evidently proportional to a_p. The curve thus obtained for a_p is a continuous function of p and approximates more closely to the value of the integral as the number of terms is increased.

To illustrate, suppose we wish to analyze a certain $\phi(x)$ into its harmonic constituents for values of x between 0 and π. Call the width of the m elements of the machine π, and set each rod on its lever B so that the distance d is proportional to the corresponding ordinate of the curve $y = \phi(x)$; in other words, set the rods so that their ends form this curve. On turning the machine, then, there will be traced on the paper a curve whose abscissas are values of γ in (92) and whose ordinates are proportional to the values of the various coefficients. To find b_5, say, we choose a value of γ equal to $\frac{5\pi}{m} = \frac{\pi}{16}$, if $m = 80$; then the ordinate for this value of γ, which is one sixteenth of the whole length of the curve in this case, is proportional to b_5. The various other coefficients are found in the same way, and the great advantage of the machine is that all the coefficients are determined by a single process.

To change the motion of the elements from the cosine to the sine form, each eccentric is turned through 90°. The accuracy of such a machine of 80 elements may be placed at about 1%.

APPLICATIONS

64. From the preceding discussion it is evident that the harmonic analyzer is of use in any case where it is desired to add a number of harmonic sine or cosine curves, or, as is more frequently the case, when some curve is to be analyzed and the amplitudes of its component harmonics determined. Tracings of sound waves can be analyzed in this way, but probably the best example of an application is in the analysis of alternating current waves. Here the simple manipulation of the machine takes the place of the more or less tedious algebraic methods usually given in works on this subject.*

PROBLEMS

1. Develop the sine series which gives $y = 0$ for x between 0 and $\frac{\pi}{2}$; and $y = c$ for x between $\frac{\pi}{2}$ and π. Plot and add the first four or five terms.

$$\left(y = \frac{2c}{\pi}\left\{\frac{\sin x}{1} - \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \frac{2\sin 6x}{6} + \cdots\right\}\right).$$

2. Do this for the corresponding cosine series.

$$\left(y = \frac{2c}{\pi}\left\{\frac{\pi}{4} - \frac{\cos x}{1} + \frac{\cos 3x}{3} - \frac{\cos 5x}{5} + \cdots\right\}\right).$$

3. Show that $x = 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right\}$, for x between 0 and π.

4. Develop $f(x)$ in a sine series if $f(x) = \frac{l}{3}$, for $x = 0$ to $\frac{c}{3}$; $f(x) = 0$, for $x = \frac{c}{3}$ to $\frac{2c}{3}$; $f(x) = -\frac{l}{3}$, for $x = \frac{2c}{3}$ to c.

$$\left(f(x) = \frac{l}{\pi}\left\{\sin\frac{2\pi x}{c} + \frac{1}{2}\sin\frac{4\pi x}{c} + \frac{1}{4}\sin\frac{8\pi x}{c} + \frac{1}{5}\sin\frac{10\pi x}{c} + \cdots\right\}\right).$$

5. Verify

$$e^x = 2c\left\{\frac{1}{2}\frac{e^c - 1}{c^2} - \frac{e^c + 1}{c^2 + \pi^2}\cos\frac{\pi x}{c} + \frac{e^c - 1}{c^2 + 4\pi^2}\cos\frac{2\pi x}{c}\right.$$
$$\left. - \frac{e^c + 1}{c^2 + 9\pi^2}\cos\frac{3\pi x}{c} + \cdots\right\} \text{ from } x = 0 \text{ to } x = c.$$

* For a simple graphical method of analysis see paper by Charles S. Slichter, "Graphical Computation of Fourier's Constants for Alternating Current Waves," *Electrical World*, July 15, 1909.

6. If $f(x) = 0$ from $x = -\pi$ to 0; and $f(x) = x$ from $x = 0$ to π, show that

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\}.$$

7. Develop $c + \sin x$ in a cosine series between 0 and π; and in a complete Fourier's series (see equation (46)) between $-\pi$ and $+\pi$.

$$\left(y = c + \frac{2}{\pi}\left\{1 - \frac{2}{3}\cos 2x - \frac{2}{15}\cos 4x - \frac{2}{35}\cos 6x + \cdots\right\};\ y = c + \sin x\right).$$

8. Outline the curve between $-\pi$ and $+\pi$, formed by the addition of series (20) and (44).

CHAPTER VII

THE LINEAR FLOW OF HEAT

65. In Chapters III–V we have already discussed a number of the simpler problems of heat flow. These have included the case of the steady state for several different conditions, and the simplest case in which the temperature varies with time, namely, the periodic flow. With the single exception of the steady state for a plane, in which we were forced to assume one of the results derived later in the study of Fourier's series, these problems could all be solved without the use of this analysis; but we now come to a class of problems, at once more interesting and more difficult, in which continual use is made of Fourier's series and integrals.

In the present chapter we shall take up a number of cases of the flow of heat in one dimension. These will include the problem of the infinite solid, in which the heat is supposed to have a given initial distribution, — that is, the initial temperature is known for every point, — and starts to flow at time $t = 0$; the so-called semi-infinite solid which has one plane bounding face, usually under a given condition of temperature; the slab with its two plane bounding faces; also the case of the long rod with radiating surface; and the problem of heat sources. In these several cases the solutions hold equally well for the one-dimensional flow of heat in an infinite solid, or for the flow along a rod whose surface, save in the fourth case above mentioned, is supposed to be impervious to heat. In all the problems discussed in this chapter, save that of the radiating rod, the solutions must first of all satisfy the Fourier conduction equation, which becomes for one dimension

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}. \qquad (1)$$

As we saw in Art. 16, this must be modified for the case of the radiating rod by the addition of a third term.

Case I

Infinite Solid. Initial Distribution of Heat given

66. Take the x-direction as that of the flow of heat. Then all planes parallel to the yz-plane will be isothermal surfaces, and the initial temperature of these surfaces is known as a function of their x-coördinates. The problem is to determine their temperatures at any subsequent time.

The solution must satisfy (1) and the condition

$$\theta = f(x) \text{ when } t = 0. \tag{2}$$

We shall solve (1) by a process which is, at the outset, the same as that employed in Art. 38, namely, the substitution in (1) of

$$\theta = e^{at+\beta x}. \tag{3}$$

This gives

$$a = h^2\beta^2. \tag{4}$$

Putting now

$$\beta = \pm i\gamma, \tag{5}$$

instead of $a = \pm i\gamma$ as before, we get

$$\theta = Le^{-h^2\gamma^2 t} \cdot e^{i\gamma x}, \tag{6}$$

and

$$\theta = Me^{-h^2\gamma^2 t} \cdot e^{-i\gamma x}. \tag{7}$$

But since

$$e^{\pm i\gamma x} = \cos \gamma x \pm i \sin \gamma x, \tag{8}$$

we get, on combination of (6) and (7) by addition or subtraction, — choosing suitable values for L and M, — the particular solutions

$$\theta = e^{-h^2\gamma^2 t} \cdot \cos \gamma x, \tag{9}$$

and

$$\theta = e^{-h^2\gamma^2 t} \cdot \sin \gamma x. \tag{10}$$

These are particular solutions of (1) for any value of γ, the latter being neither a function of x nor t. Now we can multiply these by A and B, any functions of γ, and obtain the sum of an infinite series of terms represented by

$$\theta = \int_0^\infty (A \cos \gamma x + B \sin \gamma x)\, e^{-h^2\gamma^2 t} d\gamma, \tag{11}$$

also as a solution of (1) by the proposition of Art. 8.

The functions A and B must be so determined that for $t=0$, (11) goes over into $f(x)$. Now Fourier's integral (VI, (77)) gives

$$f(x)=\frac{1}{\pi}\int_0^\infty d\gamma\int_{-\infty}^{+\infty} f(\lambda)\cos\gamma(\lambda-x)\,d\lambda, \tag{12}$$

and from (11) this must equal

$$\int_0^\infty (A\cos\gamma x+B\sin\gamma x)\,d\gamma. \tag{13}$$

Hence
$$A=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(\lambda)\cos\gamma\lambda d\lambda, \tag{14}$$

and
$$B=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(\lambda)\sin\gamma\lambda d\lambda, \tag{15}$$

and if these values are substituted in (11), we finally have

$$\theta=\frac{1}{\pi}\int_0^\infty e^{-h^2\gamma^2 t}d\gamma\int_{-\infty}^{+\infty} f(\lambda)\cos\gamma(\lambda-x)\,d\lambda. \tag{16}$$

This is then the required solution, for it satisfies (1) and reduces for $t=0$ to (12), that is, to $f(x)$. It gives the value of θ for any chosen values of x or t.

67. This equation can be simplified and put in a more useful form by changing the order of integration and evaluating one of the integrals. For

$$\theta=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(\lambda)\,d\lambda\int_0^\infty e^{-h^2\gamma^2 t}\cos\gamma(\lambda-x)\,d\gamma, \tag{17}$$

and
$$\int_0^\infty e^{-h^2\gamma^2 t}\cos\gamma(\lambda-x)\,d\gamma=\frac{1}{2h}\sqrt{\frac{\pi}{t}}\cdot e^{\frac{-(\lambda-x)^2}{4h^2 t}}, \tag{18}$$

since (see Appendix C)

$$\int_0^\infty e^{-m^2y^2}\cos ny\,dy=\frac{\sqrt{\pi}}{2m}e^{\frac{-n^2}{4m^2}}. \tag{19}$$

Hence
$$\theta=\frac{1}{2h\sqrt{\pi t}}\int_{-\infty}^{+\infty} f(\lambda)\,e^{\frac{-(\lambda-x)^2}{4h^2 t}}\,d\lambda. \tag{20}$$

By putting
$$\beta=\frac{\lambda-x}{2h\sqrt{t}},\quad\text{or}\quad\lambda=x+2h\sqrt{t}\beta, \tag{21}$$

we secure the still shorter form

$$\theta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(x + 2h\sqrt{t}\beta) e^{-\beta^2} d\beta. \tag{22}$$

We may regard this as our final solution, as it is much easier to handle than the other forms. If $f(x) = C$, a constant, then $f(x + 2h\sqrt{t}\beta) = C$, and the integral reduces to the "probability integral" (see Appendix D). If $f(x) = x^2$, say, then (22) becomes

$$\theta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (x^2 + 4hx\sqrt{t}\beta + 4h^2t\beta^2) e^{-\beta^2} d\beta. \tag{23}$$

Remembering that x is a constant as regards this integration, these three integrals can be readily evaluated (see Appendixes B, C, and D). Also for many other forms of $f(x)$ the integration is not difficult.

68. If $f(x)$ is of more than one form, or possesses discontinuities, it may be necessary to split the integral (22) into two or more parts. For example, suppose that $f(x) = \theta_0$ between the limits $x = l$ and $x = m$, and that $f(x) = 0$ outside these limits — a condition which would correspond to the sudden introduction of a slab at temperature θ_0 between two infinite blocks of the same material and at zero temperature. We write the integral (22)

$$\theta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{a} 0 \cdot e^{-\beta^2} d\beta + \frac{1}{\sqrt{\pi}} \int_{a}^{b} \theta_0 e^{-\beta^2} d\beta + \frac{1}{\sqrt{\pi}} \int_{b}^{\infty} 0 \cdot e^{-\beta^2} d\beta. \tag{24}$$

In determining the limits a and b it must be remembered that x (as well as t) is a constant for each particular evaluation of the integral, and that the initial temperature condition is really expressed as a function of the variable of integration λ, that is, $\theta_0 = f(\lambda)$. The limits a and b will then be the values of β corresponding to $\lambda = l$ and $\lambda = m$; and from (21) these are seen to be $\frac{l - x}{2h\sqrt{t}}$ and $\frac{m - x}{2h\sqrt{t}}$ respectively. Equation (24) then reduces to

$$\theta = \frac{\theta_0}{\sqrt{\pi}} \int_{\frac{l-x}{2h\sqrt{t}}}^{\frac{m-x}{2h\sqrt{t}}} e^{-\beta^2} d\beta. \tag{25}$$

This solution may be readily applied to the case in which $f(x) = \theta_0$ for $x > 0$, and $f(x) = 0$ for $x < 0$, for in this event the limits are seen at once to be $-\dfrac{x}{2h\sqrt{t}}$ and $+\infty$.

APPLICATIONS

69. Concrete Wall. While perhaps not having the variety of applications which we shall find for Case II, next to be considered, the foregoing equations admit of the solution of many interesting problems. For example, suppose a concrete wall 60 cm. (23.6″) thick is to be formed by pouring concrete in a trench cut in soil at a temperature of $-4°$ C. (24.8° F.), the concrete being poured at 8° C. (46.4° F.). It is desired to know how long before the freezing temperature will penetrate the wall to a depth of 5 cm. (2″). In other words, will the wall as a whole have time to "set" before it is frozen?

To apply the foregoing equations we must first assume that the soil has the same diffusivity as the concrete, as would be approximately true in many cases, and that latent-heat considerations can be neglected. The solution then follows at once from the equation of the last article. Taking the origin at the center of the wall, we have $l = -30$ cm., $m = +30$ cm., and $x = \pm 25$ cm. Choosing, say, the positive value for x, and shifting our temperature scale so that the initial soil temperature is brought to zero, while the freezing temperature becomes 4° and the initial wall temperature 12°, (25) becomes

$$4 = \frac{12}{\sqrt{\pi}} \int_{\frac{-55}{2h\sqrt{t}}}^{\frac{5}{2h\sqrt{t}}} e^{-\beta^2} d\beta.$$

To find t we must determine the limit $q \left(= \dfrac{5}{2h\sqrt{t}}\right)$ so that

$$\frac{2}{\sqrt{\pi}} \int_{-11q}^{q} e^{-\beta^2} d\beta \left(\equiv \frac{2}{\sqrt{\pi}} \int_0^q e^{-\beta^2} d\beta + \frac{2}{\sqrt{\pi}} \int_0^{11q} e^{-\beta^2} d\beta \right) = \frac{2}{3}.$$

With the aid of the probability integral curve we readily find q to be about .055, which gives

$$t = \left(\frac{5}{.11\,h}\right)^2 = \frac{25}{.0121 \times .0058} = 356{,}000 \text{ sec.} = 4.1 \text{ days.}$$

70. It may be remarked that in solving this problem we have also accomplished the solution of another which, at first sight, appears by no means identical with it. Suppose the same temperature conditions to exist, but the wall to be only half as thick, and one face in contact, not with earth, but with some material practically impervious to heat, or at least a very much poorer conductor than cement; for example, cork or concrete forms of dry wood. To see the similarity of the two problems, notice that in the first one conditions of symmetry* show that there would be no transference of heat across a middle plane in the wall; hence this plane could be made of material impervious to heat without altering the conditions. We could then remove half of the wall without affecting the half on the other side of this impermeable plane, in which case we should have our present problem.

71. In the above solutions we have omitted consideration of three important factors which would generally be present in any practical case, and which would serve to retard to a considerable extent the freezing of the wall. These are the latent heat of freezing of the water of the concrete, the heat of reaction which accompanies the setting of concrete, and the insulating effect of wooden forms which are frequently used for such a wall. The theoretical treatment of these factors would be beyond the aims of the present work.

72. Thermit Welding. As a further application let us take another and more difficult problem. Suppose two sections of a steel shaft 30 cm. (11.8″) in diameter are to be welded end to end by the thermit process. The crevice between the ends is 8 cm. (3.1″) wide, and the pouring temperature of the molten steel is assumed to be about 3000° C., while the shaft is heated to 500° C. (that is, some "preheating"). It is found that a temperature much above 700° C. (the "recalescence point") modifies to some extent the character of the steel of the shaft, and it is desired to know, then, to what depth this temperature will penetrate, or, in other words, how far back from the ends this overheating will extend.

* It is to be noted that this point of view demands a temperature condition symmetrical about the middle plane of the wall. That this is satisfied in the present case, *that is*, $f(\lambda) = \theta_0$, *a constant*, is evident.

We shall attempt only an approximate solution of this problem, neglecting any changes which the thermal constants undergo at higher temperatures, also radiation losses and other complicating factors; and shall interpret it as that of the introduction of a "slab" of steel at 3000° C. between two infinite masses of steel at 500° C. Taking the origin in the middle and putting $l = -4$ and $m = +4$, (25) becomes, after shifting the temperature scale 500°,

$$200 = \frac{2500}{\sqrt{\pi}} \int_{\frac{-4-x}{2h\sqrt{t}}}^{\frac{4-x}{2h\sqrt{t}}} e^{-\beta^2} d\beta.$$

Our problem is then to find the largest value of x which will satisfy the above relation, that is, which will afford a value of the probability integral equal to $\frac{200}{1250}$, or .16.

We can most conveniently arrive at a solution by the method of "trial and error." Thus, if $x = 5$, that is, 1 cm. from the original end of the shaft, the limits of the above integral may be called $-9q$ and $-q\left(q \equiv \frac{1}{2h\sqrt{t}}\right)$, and a little inspection of the table in Appendix D shows that to give the integral the value .16, q must be either .018 or .994. For $x = 10$ the limits are $-14q$ and $-6q$, which necessitates q being either .019 or .165; and a few more trials show that if $x = 24.5$, with corresponding limits of $-28.5q$ and $-20.5q$, there is only a single value to be found for q, and this is approximately equal to .027.

This, then, is the key to the solution, for the second and larger of the two q values in the above pairs will evidently give the shorter time, or, in other words, the time at which the point first reaches this temperature. For the smaller values of x the temperature goes higher than this value of 700° and later falls to this point at a time afforded by the first value of q. When the two values are just equal it means that the temperature just reaches this value, and the time in this case will be given by

$$t = \frac{1}{4h^2q^2} = \frac{1}{4 \times .121 \times .027^2} = 2830 \text{ sec.}$$

The overheating* then extends in to 20.5 cm. (8.1″) from the end and reaches this point in 47 min.

73. It is well to note in these, as in any other applications, how the results would be affected by changes in the conditions which enter. In the first case, for instance, it is readily seen that the time will come out the same for any two temperatures of the soil and concrete which have the same ratio; for example, $-2°$ and $+4°$, or $-15°$ and $+30°$. Moreover a consideration of the limits shows that the time is inversely proportional to the diffusivity h^2. In the last illustration this same inverse proportionality of time and diffusivity also holds, and we can in addition draw the rather striking conclusion that the *depth* to which a given temperature will penetrate under such conditions is independent of the thermal constants of the medium.† The *time* it takes to reach this depth, however, depends, as just mentioned, on the diffusivity.

PROBLEMS

1. Show that if the initial temperature is everywhere θ_0, a constant, the temperature must always be θ_0.

In this case $$\theta = \frac{\theta_0}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\beta^2} d\beta = \theta_0.$$

(See Appendix D for values of the probability integral.)

2. Show that if θ is initially equal to x, it must always be equal to x; and if it is initially equal to x^2, it will be $x^2 + 2h^2t$ at any time later.

3. In the application of Art. 69 determine when the freezing temperature would reach the center of the wall. (4.8 days.)

4. A slab of molten lava at 1000° and 40 m. thick is intruded in the midst of rock at about 0°. What will be the temperatures at the center and edges of the slab after cooling for 1 day and for 100 years? (Use $h^2 = .0118$ for both lava and surrounding rock.)

(Center, 1000° and 183°; edge, 500° and 178°.)

5. Frozen soil at $-6°$ C. is to be thawed by spreading over the surface a 15 cm. layer of hot ashes and cinders at 800° C. and then covering the

* Whether or not such heating is injurious to the material depends largely on the rate of cooling. This in practice is slow, so as to anneal the material.

† This is only true, of course, when the heated material introduced is of the same *character as the body itself.*

surface of this layer to prevent heat loss. Assuming that the diffusivity of soil and ashes may be taken at .0049, and that the latent heat of fusion of the water content may be taken account of by supposing that the soil has to be raised to, say, 5° instead of merely to zero, to produce melting, how far will the thawing proceed in half a day?

SUGGESTION. Try $x = 50$ cm., 60 cm., etc. Note that the problem is equivalent to that for a slab of twice the thickness, with ground on each side. (45 cm., or $x = 60$ cm.)

6. A bar l cm. long, in which the temperatures have assumed a steady state with one end at 0° and the other at 100°, is placed in end-to-end contact between two very long similar bars at 0°. It is assumed that the surfaces of the bars are protected from loss of heat, and the origin is taken at the zero end of the middle bar. Work out the formula for the temperature at any point of the bar and apply it to the case of an iron bar ($h^2 = .173$) 100 cm. long after 15 min. of cooling. Find the temperatures at the center, at the hot end, and at the cold end. (49.75°, 42.95°, 7.05°.)

7. A great pile of dry soil ($h^2 = .0031$) at $-30°$ C. is deposited on similar soil at $+2°$ C. How long will it take the zero temperature to penetrate to a depth of 1 m.? (7.9 days.)

8. In the application of Art. 72 compute the distance to which the temperature 1300° C. will penetrate.* (2 cm.)

CASE II

Semi-infinite Solid with One Plane Bounding Face at Constant Temperature. Initial Distribution of Heat given

74. This is the case of the body extending to infinity in the positive x-direction only, and bounded by the yz-plane, which is kept at a constant temperature. The temperature for every point (plane) of the body is given for the time $t = 0$.

75. Boundary at Zero Temperature. We have here to seek a solution of

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}, \tag{1}$$

subject to the conditions $\theta = 0$ at $x = 0$, (26)

and $\theta = f(x)$ when $t = 0$. (27)

* Waterhouse (*Proc. Am. Soc. Testing Materials*, 6, p. 248 (1906)), finds that for 1% carbon nickel steel a temperature above 1300° C. results in "burning" or injuring the metal; hence the application of the above problem.

It is possible to treat this case as a special form of I (Arts. 66–68) by imagining that for every positive (or negative) temperature at distance x there is an equal negative (or positive) temperature at distance $-x$. In other words, if there should be a distribution of heat on the side of the negative x identical with, but opposite in sign to, that on the positive side, the flow of heat would be such as to continually keep the temperature of the yz-plane zero. A little thought on the symmetry of such a temperature distribution will suffice to show that this conclusion is sound; for there is no more reason for the boundary surface to take positive temperatures under these conditions than negative, and hence its temperature will be zero.

To express this condition mathematically, let us suppose that for points on the positive side of the origin $\lambda = \lambda_1$, and on the negative side $\lambda = -\lambda_2$. Then λ_1 and λ_2 are each essentially positive, and the temperature $(f(\lambda))$ can be expressed as $f(\lambda_1)$ for the positive region, and $-f(\lambda_2)$ for the negative. Equation (20) can then be written for this case

$$\theta = \frac{1}{2h\sqrt{\pi t}}\left\{\int_0^\infty f(\lambda_1)\, e^{\frac{-(\lambda_1 - x)^2}{4h^2t}}\, d\lambda_1 \right.$$
$$\left. + \int_\infty^0 -f(\lambda_2)\, e^{\frac{-(-\lambda_2 - x)^2}{4h^2t}}\, (-d\lambda_2)\right\}, \qquad (28)$$

the lower limit of the second integral being $+\infty$ instead of $-\infty$, as it would be if λ were the variable. But since the value of a definite integral is independent of the variable of integration (cf. Arts. 57 and 60), we can substitute λ (or any other symbol) for λ_1 and λ_2 in the above equation, which can then be reduced to

$$\theta = \frac{1}{2h\sqrt{\pi t}}\int_0^\infty f(\lambda)\left\{e^{\frac{-(\lambda - x)^2}{4h^2t}} - e^{\frac{-(\lambda + x)^2}{4h^2t}}\right\} d\lambda. \qquad (29)$$

Making substitutions similar to (21), namely,

$$\beta = \frac{\lambda - x}{2h\sqrt{t}}, \text{ and } \beta' = \frac{\lambda + x}{2h\sqrt{t}},$$

this becomes $$\theta = \frac{1}{\sqrt{\pi}}\left\{\int_{\frac{-x}{2h\sqrt{t}}}^{\infty} f(2\beta h\sqrt{t}+x)e^{-\beta^2}d\beta\right.$$

$$\left.-\int_{\frac{+x}{2h\sqrt{t}}}^{\infty} f(2\beta' h\sqrt{t}-x)e^{-\beta'^2}d\beta'\right\}, \tag{30}$$

or, what amounts to the same thing,

$$\theta = \frac{1}{\sqrt{\pi}}\left\{\int_{\frac{-x}{2h\sqrt{t}}}^{\infty} f(2\beta h\sqrt{t}+x)e^{-\beta^2}d\beta\right.$$

$$\left.-\int_{\frac{+x}{2h\sqrt{t}}}^{\infty} f(2\beta h\sqrt{t}-x)e^{-\beta^2}d\beta\right\}. \tag{31}$$

It is well to assure ourselves that (31) is the required solution. From the manner of its formation, that is, originally from (9) and (10), it must be a solution of (1), while for $x=0$ the two integrals are evidently equal and opposite in sign, so condition (26) is fulfilled. As to condition (27) we see that for $t=0$ the second integral vanishes, and the whole expression reduces to

$$\theta = f(x)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\beta^2}d\beta = f(x). \tag{32}$$

76. An interesting special case is that in which the initial temperature is θ_0 throughout the body except at the yz-surface, which is still kept at zero. $f(\lambda)\,(=f(2\beta h\sqrt{t}+x)$ or $f(2\beta h\sqrt{t}-x))$ then reduces to θ_0, so that (31) becomes

$$\theta = \frac{\theta_0}{\sqrt{\pi}}\left\{\int_{\frac{-x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2}d\beta - \int_{\frac{+x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2}d\beta\right\} \tag{33}$$

$$= \frac{\theta_0}{\sqrt{\pi}}\int_{\frac{-x}{2h\sqrt{t}}}^{\frac{+x}{2h\sqrt{t}}} e^{-\beta^2}d\beta \tag{34}$$

$$= \frac{2\,\theta_0}{\sqrt{\pi}}\int_{0}^{\frac{x}{2h\sqrt{t}}} e^{-\beta^2}d\beta, \tag{35}$$

since $e^{-\beta^2}$ is an *even* function (Art. 60).

77. Boundary at θ_0; Initial Temperature of Body Zero. By an extension of (35) we can handle this case at once. For if (35) is written for a negative initial temperature $-\theta_0$, we have

$$\theta_1 = -\frac{2\,\theta_0}{\sqrt{\pi}}\int_0^{\frac{x}{2h\sqrt{t}}} e^{-\beta^2}d\beta\,; \tag{36}$$

and if θ_0 is then added to each side, we get

$$\theta \equiv \theta_1 + \theta_0 = \theta_0\left\{1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2h\sqrt{t}}} e^{-\beta^2}d\beta\right\} \tag{37}$$

$$= \frac{2\,\theta_0}{\sqrt{\pi}}\int_{\frac{x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2}d\beta. \tag{38}$$

This process is, of course, merely equivalent to shifting the temperature scale, as we have had frequent occasion to do in previous problems.

78. Law of Times. An interesting fact can be deduced from (35) and (38), for it is easily seen that any particular temperature θ is attained at distances x_1 and x_2 from the boundary surface in times t_1 and t_2 conditioned by the relation

$$\frac{x_1}{2\,h\sqrt{t_1}} = \frac{x_2}{2\,h\sqrt{t_2}}, \tag{39}$$

or

$$\frac{t_1}{t_2} = \frac{x_1^2}{x_2^2}. \tag{40}$$

This gives the law that *the times required for any two points to reach the same temperature are proportional to the squares of their distances from the boundary plane,* a statement which is true whether the body is initially at a uniform temperature and the surface at zero, or initially at zero and the surface heated, provided only that the surface keeps its temperature constant in each case.

It can also be at once deduced that the time required for any point to reach a given temperature is inversely proportional to the diffusivity h^2. Both these relations are of wide application, and the one or the other of them holds good for a large number of cases of heat conduction. We have already noted a case in which the second law holds in Art. 73.

79. Rate of Flow of Heat. We can now determine the rate at which heat flows into or out of a body through any unit of area of plane surface parallel to the boundary. To do this write (38) as a function of the limits of integration,

$$\theta = \frac{2\,\theta_0}{\sqrt{\pi}}\left[\phi(\infty) - \phi\left(\frac{x}{2\,h\sqrt{t}}\right)\right]. \tag{41}$$

Then differentiate,

$$\frac{\partial\theta}{\partial x} = \frac{2\,\theta_0}{\sqrt{\pi}}\left[0 - e^{\frac{-x^2}{4h^2t}}\frac{1}{2\,h\sqrt{t}}\right] \text{ (see Appendix F)} \tag{42}$$

$$= -\frac{\theta_0}{h\sqrt{\pi t}}e^{\frac{-x^2}{4h^2t}}. \tag{43}$$

The rate of flow of heat through any unit area of surface parallel to the yz-boundary plane is then

$$W = -k\frac{\partial\theta}{\partial x} = \frac{k\theta_0}{h\sqrt{\pi t}}e^{\frac{-x^2}{4h^2t}}, \tag{44}$$

or for the boundary plane $x = 0$

$$W = \frac{k\theta_0}{h\sqrt{\pi t}}. \tag{45}$$

The same expressions, save with a negative sign, hold for the case of the boundary at zero.

80. Temperature of Surface of Contact. Suppose two infinite bodies A and B of conductivities and diffusivities k_1, h_1 and k_2, h_2 respectively, each with a single plane surface and with these surfaces placed in contact. Assume that A and B are initially at temperatures θ_1 and θ_2 respectively, and imagine for the moment that the boundary surface is kept, either by the continuous addition or subtraction of heat, at the constant temperature θ_0, where $\theta_1 > \theta_0 > \theta_2$. We shall determine what conditions must be fulfilled that this surface of contact may receive as much heat from one body as it loses to the other, and hence will require no gain or loss of heat from the outside to keep constantly at θ_0; in other words, we shall determine this temperature of the surface of contact.

Each unit of area of surface of contact receives heat from A at the rate

$$W_1 = \frac{k_1(\theta_1 - \theta_0)}{h_1\sqrt{\pi t}}, \tag{46}$$

while it loses to B at the rate

$$W_2 = \frac{k_2(\theta_0 - \theta_2)}{h_2\sqrt{\pi t}}. \tag{47}$$

Then if these two are equal, the boundary plane will neither gain nor lose heat permanently and hence will remain constant in temperature, so

$$\frac{k_1(\theta_1 - \theta_0)}{h_1} = \frac{k_2(\theta_0 - \theta_2)}{h_2}, \tag{48}$$

or

$$\theta_0 = \frac{\left\{\frac{k_1\theta_1}{h_1} + \frac{k_2\theta_2}{h_2}\right\}}{\left\{\frac{k_1}{h_1} + \frac{k_2}{h_2}\right\}}. \tag{49}$$

If $k_1 = k_2$ and $h_1 = h_2$, $\theta_0 = \frac{\theta_1 + \theta_2}{2}$, as we should expect.

APPLICATIONS

81. Concrete. In a fire test the surface of a large mass of concrete was heated to 700° C. (1292° F.); how long should it take the temperature of 100° C. (212° F.) to penetrate 30 cm. (11.8″) if the initial temperature of the mass was 20° C. (68° F.)? From (38) we have

$$80 = 680\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2}d\beta,$$

or $t = 31{,}500$ sec. $= 8.8$ hr. For 300° C. (572° F.) the time turns out to be 32 hr.

82. Soil. How far will the freezing temperature penetrate in 24 hr. in soil ($h^2 = .0049$) at 5° C. (41° F.) if the surface is lowered to -10° C. (14° F.)?

$$-5 = -15\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2}.d\beta;\ x = 28.2 \text{ cm. } (11.1'').$$

For twice this depth it will take 4 days, etc.

If the initial temperature of soil is 2° C. (35.6° F.) and the surface is cooled to – 24° C. (– 11° F.), how long before the temperature will fall to zero at the depth of a meter?

$$-2 = -26 \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2} d\beta; \; t = 326{,}000 \text{ sec.} = 3.8 \text{ days.}$$

As no account has been taken of the latent heat of freezing for the moisture of the soil in the last two problems, the distance in the first problem is undoubtedly too large, and the time in the second too small, for the actual case. Even in the case of concrete, unless it is old and thoroughly dry, there is a considerable lag in the heating effect as the boiling point is passed,* showing latent-heat effects.

An exact treatment of these latent-heat considerations must be reserved for Chapter IX, but in the following problem an approximate solution for a particular case is suggested.

83. The Thawing of Frozen Soil. Soil at – 6° C. (21° F.), of diffusivity .0049 and moisture content 3%, is to be thawed by heating the surface with a coke fire to 800° C. (1472° F.). The question is: How far will the thawing proceed in a given time?

To take account of the latent heat of fusion of the 3% moisture we will note that, since the specific heat of such soil is taken as .45 (undoubtedly, however, this is a rather high figure for such small moisture content), the heat required to thaw this moisture per gram of soil would be the same as that which would raise this soil .03 × 80 ÷ .45, or about 5° in temperature. This is nearly equivalent to saying that the soil must be raised to 5° C. (41° F.) to produce thawing, that is, a total rise of 11° C. Then

$$11 = 806 \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2} d\beta,$$

and we find that

$$\frac{x}{2h\sqrt{t}} \text{ must be about } 1.74, \text{ or } t = \frac{x^2}{.0595} = 16.8\,x^2.$$

* See Woolson, *Proc. Am. Soc. for Testing Materials*, 6, p. 441 (1906).

Then for a thawing of 45 cm. (1.5′), $t = 34{,}000$ sec., or 9.5 hr.; and for 90 cm. (3′), 38 hr., etc.

While local conditions (varying diffusivities and moisture contents) would alter these figures considerably, the law that the time for thawing would vary as the square of the depth holds good in any case in which the soil is initially at sensibly the same temperature throughout. If it is not as cold below, the thawing will proceed faster than this law would indicate.

84. "Shrunk" Fittings. As a problem of a somewhat different type from the preceding let us consider the thermal principles involved in the removal by heating of a ring or collar which has been "shrunk" on to a cylinder or wheel. If the thickness is small compared with the diameter, it may be treated as a case of one-dimensional transmission, and as a very good example we may cite the case of the locomotive tire. Suppose such a tire 7.62 cm. (3″) thick is to be removed by heating its outer surface; let us question at what time the differential expansion of tire and rim would be a maximum and hence the tire be most readily removed. We shall assume that this differential expansion is determined by the magnitude of the temperature gradient across the boundary of tire and rim.

From (43) $$\frac{\partial\theta}{\partial x} = -\frac{\theta_0}{h\sqrt{\pi t}}\,e^{\frac{-x^2}{4h^2t}}.$$

To find when this is a maximum, differentiate with respect to t and equate to zero. Then

$$t = \frac{1}{2}\frac{x^2}{h^2}. \tag{50}$$

$$\therefore \left(\frac{\partial\theta}{\partial x}\right)_{\text{max.}} = \frac{-\theta_0}{h\sqrt{\pi t e}}. \tag{50 a}$$

So in this case ($h^2 = .121$) $t = 240$ sec., or 4 min.

The above discussion of the problem is based on the conditions of Art. 77, namely, for the surface heated suddenly to the temperature θ_0, as by immersion in a bath of molten metal. As a *matter of fact* the surface heating in the practical case would

generally be a more gradual process, brought about in many cases by a gas flame. A rigorous solution of this complicated problem is very difficult, but the following is offered as being a good approximate solution. Imagine in the case of the locomotive tire just considered that 5 cm. thickness is added to the tire and that the outer surface is, as before, suddenly raised to temperature θ_0. The temperature of the original surface will then

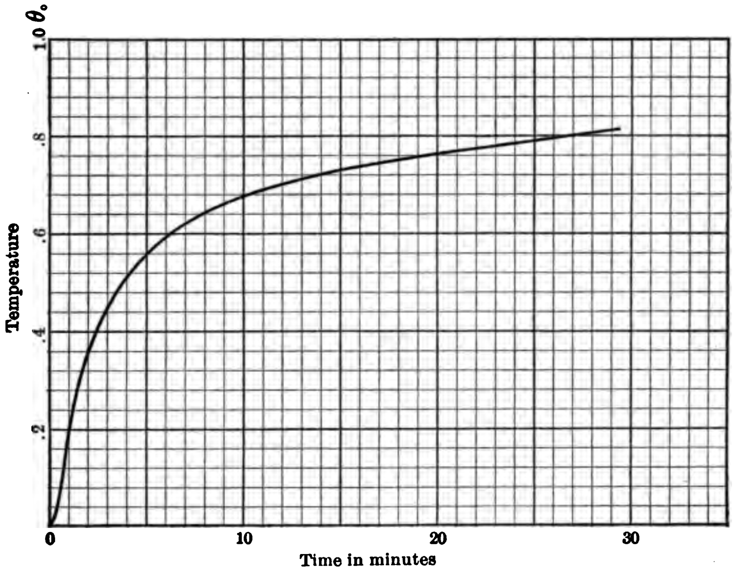


FIG. 10. A type of theoretical temperature-time curve obtained on the assumption of Art. 84

The more nearly the actual heating curve of the surface approaches this type, the better can the case be handled theoretically

be given by (38) and will be found to rise gradually (see Fig. 10), increasing more rapidly at first and more slowly later, just as would be the case if this surface was flame heated. By varying the thickness of metal which we are to assume added (the 5 cm. added in this case yields a very plausible curve) and plotting the temperature-time curve as in Fig. 10 for each case, a result may be obtained very nearly like the actual heating conditions.

The problem is then reduced to the preceding, save that the tire is imagined to be 5 cm. thicker. The time comes out 11 min. For a slower rate of heating the time would be correspondingly longer.

A point of interest in this connection is a comparison of the actual maximum temperature gradients for the rapid and slow

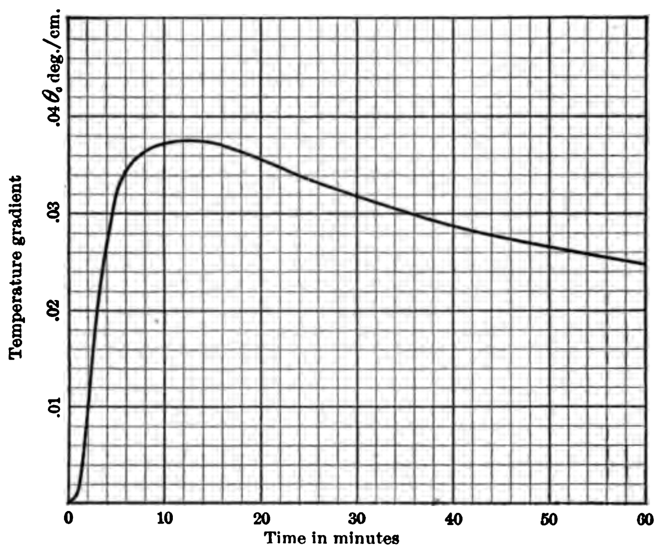


FIG. 11. Curve showing the variation of temperature gradient with time, at a distance of 12.6 cm. (5″) below a surface of steel suddenly heated to θ_0; or 7.6 cm. (3″) below a surface heated as in Fig. 10

The best time to attempt the removal of the fitting would be when the gradient is a maximum

heating, for these are the measure of the ease — or the possibility — of removal of such a shrunk fitting. Putting $t = 240$ sec. in (50 a), we get $\left(\frac{\partial \theta}{\partial x}\right)_{\max.} = .064\,\theta_0$ deg./cm., while for $t = 660$ sec. (which is the case for the maximum gradient under the slower heating (see Fig. 11)), the gradient is only $.038\,\theta_0$ deg./cm. This shows that when difficulty is expected in the removal of any shrunk-on collar, the surface heating should be done as quickly as possible — perhaps with the use of molten metal or even thermit. The above calculations would also serve to show

the time for which it is desirable to continue this heating. From the shape of the curve in Fig. 11 it is evident that it is much better to continue the heating too long than to cut it too short.

The considerations of this article would also apply to the so-called "thermal test" of car wheels, which consists in heating the rim of the wheel with molten metal for a given time. The temperature gradient might reasonably be taken as a measure of the stresses introduced in this way, and it could be determined at once from (43).

85. Hardening of Steel. A large ingot of steel ($h^2 = .121$) at θ_1 has its surface suddenly chilled to θ_2. Discuss the rate of cooling as a function of the time and of the depth in the metal.

We shall first put $\theta_0 = \theta_1 - \theta_2$ (that is, shift the temperature scale) and then get the rate of cooling by differentiating (35) just as we did (38) in Art. 79, save that now the differentiation is performed with respect to t. Then

$$\frac{\partial\theta}{\partial t} = \frac{2\,\theta_0}{\sqrt{\pi}}\left\{e^{\frac{-x^2}{4h^2t}}\cdot\left(\frac{-x}{4\,ht^{\frac{3}{2}}}\right) - 0\right\} \tag{51}$$

$$= \frac{-\theta_0 x}{2\,h\sqrt{\pi}t^{\frac{3}{2}}}\cdot e^{\frac{-x^2}{4h^2t}}, \tag{52}$$

which is the formula from which the curves of Fig. 12 have been computed for depths of .3 cm. and 1 cm. To apply to a specific problem let us question what are the rates of cooling at these depths if the initial temperature is 800° C. (1472° F.) and the chilling temperature 20° C. (68° F.), the times being chosen as those at which the metal is just cooling below the recalescence point (about 700° C. or 1292° F.).

To find the times, we put from (35)

$$680 = 780\,\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2h\sqrt{t}}} e^{-\beta^2}d\beta,$$

which gives $t = .16$ sec. for $x = .3$ cm. (.12″), and $t = 1.8$ sec. for $x = 1$ cm. (.39″). From (52) or from the curves we then find the rates of cooling to be 920 and 82 centigrade degrees per second respectively (1656 and 148 Fahrenheit degrees).

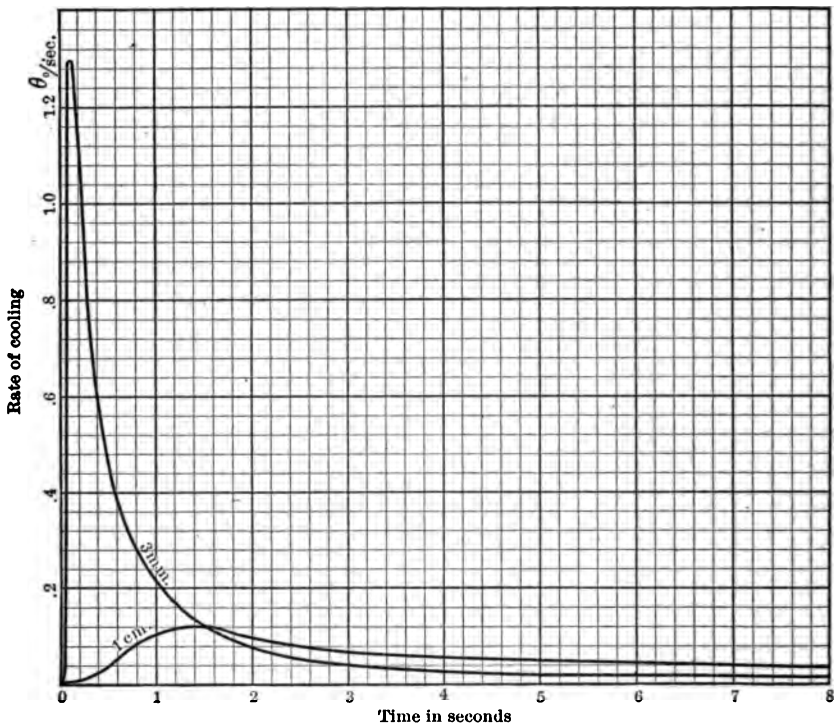


FIG. 12. Curves showing rates of cooling at depths of 3 mm. (.11″), and 1 cm. (.39″), below the surface of a steel ingot which is suddenly chilled

While it might be impossible in practice to attain as sudden a chilling of the surface as the above theory supposes, the curves of Fig. 12 will still serve to give a qualitative explanation of a well-known fact, namely, that the deeper it is desired to have the metal hardened, the hotter it must be before quenching; but that a comparatively small proportional increase in the initial temperature may produce a considerable increase in the depth of the hardening. To explain this it must be noted that one of the principal factors in hardening is the rate of cooling past the recalescence point. Now from the curves it may be seen that this rate increases to a maximum and then falls off again; hence for maximum hardness at any given depth the initial temperature

should, if possible, be high enough so that the recalescence point will not be passed until the rate of cooling has reached its maximum value.

The rapid chilling of large ingots introduces temperature stresses which frequently result in cracks. Taking the temperature gradient as a measure of this tendency to crack, the subject might be studied theoretically with the equations of the last article.

86. Cooling of Lava. We now turn to some applications of a geological nature, the first of which is the cooling of lava under water. Suppose a thickness of, say, 20 m. of lava at θ_0 (about 1000° C.) is flowed over rock at zero and immediately covered with water, — perhaps it is ejected under water, — what will be its rate of cooling?

As the water will soon cool the surface at least well below the boiling point, the problem is that of the cooling of a semi-infinite medium with boundary at zero and initial temperature conditions of θ_0 as far as $x = l$, and zero from there on to infinity. Formula (33) is for the case where the initial condition is θ_0 to infinity, and we may use it by splitting each integral into two, according to the principles explained in Art. 68, the second integral vanishing in each case, since the initial temperature for it would be zero. We have as the formula, then,

$$\theta = \frac{\theta_0}{\sqrt{\pi}}\left[\int_{\frac{-x}{2h\sqrt{t}}}^{\frac{l-x}{2h\sqrt{t}}} e^{-\beta^2}d\beta - \int_{\frac{+x}{2h\sqrt{t}}}^{\frac{l+x}{2h\sqrt{t}}} e^{-\beta^2}d\beta\right]. \tag{53}$$

Putting Kelvin's value of $h^2 = .0118$ for both lava and underlying rock, the accompanying curves (Fig. 13) are computed for $l = 20$ m. From the relationship between x and t in the above limits we readily conclude that these same curves apply to a layer n times as thick if the *times* are taken n^2 times as large, and the *distances* n times as large.

87. The Cooling of the Earth. The problem of the cooling of the earth and the estimate of its age based on such cooling has been discussed by Kelvin * and others † as a special case of the

* *Math. and Phys. Papers*, III, p. 295; *Smithsonian Report*, 1897, p. 337.

† For a good résumé of the subject see Becker, *Smithsonian Miss. Coll.* v. 56, No. 6, June, 1910.

solid with one plane bounding face; for it has been shown that the error introduced in neglecting the curvature is quite negligible. Geologically speaking, the age of the earth is counted from the epoch of Leibnitz's *consistentior status*, when the globe, or rather the crust, had attained a "state of greater consistency"

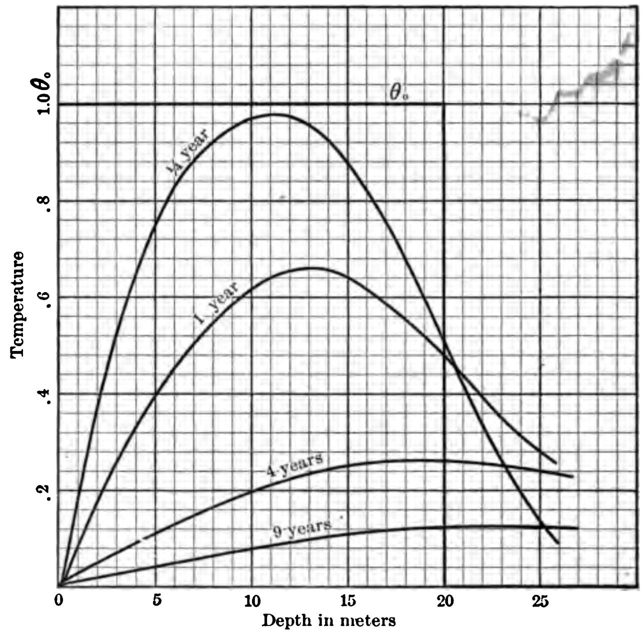


FIG. 13. Temperature curves for a layer of lava 20 meters thick, after cooling under water for various times

and the formation of the oceans became possible. Kelvin's assumption for this state was an earth whose temperature was in round numbers 3900° C. (7000° F.) throughout. He took the average value of the diffusivity as .01178 * C.G.S., and of the present surface gradient of temperature as 1° C. in 27.76 m. The problem is then to find how long it would take for the earth at the assumed initial temperature, and with the surface at a constant temperature approximately zero, to cool until the

* A. N. Winchell thinks the values $k = .0045$ and $h^2 = .0064$ are better average values for the whole earth, while Becker uses $h^2 = .00786$.

geothermal gradient at the surface has its present measured value, namely, 1° in 27.76 m.

Differentiate (35), first writing it as a function of the limits of integration:

$$\frac{\partial\theta}{\partial x}=\frac{2\,\theta_0}{\sqrt{\pi}}\frac{\partial}{\partial x}\left\{\phi\left(\frac{x}{2\,h\sqrt{t}}\right)-\phi(0)\right\} \tag{54}$$

$$=\frac{2\,\theta_0}{\sqrt{\pi}}\left\{e^{\frac{-x^2}{4\,h^2 t}}\cdot\frac{1}{2\,h\sqrt{t}}-0\right\} \tag{55}$$

$$=\frac{\theta_0}{h\sqrt{\pi t}}\,e^{\frac{-x^2}{4\,h^2 t}}, \tag{56}$$

and when $x=0$

$$\left(\frac{\partial\theta}{\partial x}\right)_0=\frac{\theta_0}{h\sqrt{\pi t}}, \tag{57}$$

or

$$t=\frac{\theta_0^2}{\pi h^2\left(\frac{\partial\theta}{\partial x}\right)_0^2}. \tag{58}$$

Putting in the constants given above, Kelvin got a value of 100,000,000 years for the age of the earth, but because of the uncertainty of the assumptions and data he placed the limits at 20–400 million years, later modifying them to 20–40 million years.

88. If the initial temperature of the earth, that is, its temperature condition at the consistentior status, instead of being uniform throughout, increased with the depth, obeying the law *

$$\theta=f(x)=mx+S, \tag{59}$$

where S is the initial surface temperature and m the initial gradient, we can solve the problem with the aid of (31); for substitution of (59) in this gives, after some simplification,

$$\theta=mx+S\cdot\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\,h\sqrt{t}}}e^{-\beta^2}d\beta. \tag{60}$$

Differentiating,

$$\frac{\partial\theta}{\partial x}=m+\frac{S}{h\sqrt{\pi t}}\,e^{\frac{-x^2}{4\,h^2 t}}. \tag{61}$$

$$\therefore\ \frac{h^2\pi}{S^2}\left(\frac{\partial\theta}{\partial x}-m\right)^2 t=e^{\frac{-x^2}{2\,h^2 t}}. \tag{62}$$

* Barus, United States Geological Survey, *Bull. No. 103*, p. 55 (1893).

When m and x are zero, this reduces, as it should, to Kelvin's solution (58). As it stands, (62) affords a value for the age of the earth, t, in terms of the geothermal gradient $\frac{\partial \theta}{\partial x}$ at any depth x, under the conditions that the initial temperature of the earth increased uniformly toward the center from some value S at the surface, and that since that time the surface has been kept at the constant temperature zero.

89. Effect of Radioactivity on the Cooling of the Earth. Since the discovery of the continuous generation of heat by disintegrating radioactive compounds, much speculation has been indulged in as to the possible effect of such heat on the earth's temperature.* Surface rocks show traces of radioactive materials, and while the quantities thus found are very minute, the aggregate amount is sufficient, if scattered with this density throughout the earth, to supply, many times over, the present yearly loss of heat. In fact, so much heat could be developed in this way that it has been practically necessary to make the assumption that the radioactive materials are limited in occurrence to a surface shell only a few kilometers in thickness.

While a satisfactory mathematical treatment of this problem is impossible with the meager data now available, it can be seen at once that radioactivity would tend to retard the cooling of the earth and hence increase our estimate of its age. A rough idea of the extent to which this is true may be had by assuming that one fourth of the present annual loss of heat is due to this cause, and that the radioactive substances are contained in a very thin outer shell. The geothermal gradient at the bottom of this shell will then be only three fourths of its observed value on the surface, because only three fourths of the heat which passes out from the earth crosses the lower surface. Then, since from (62) the age of the earth is inversely proportional to the square of the present gradient at $x = l$, the depth of the radioactive shell (if $m = 0$, and l is small), this would nearly double the calculated age of the earth.

* Becker, *Bull. Geol. Soc. of America*, **19**, p. 113 (1908).

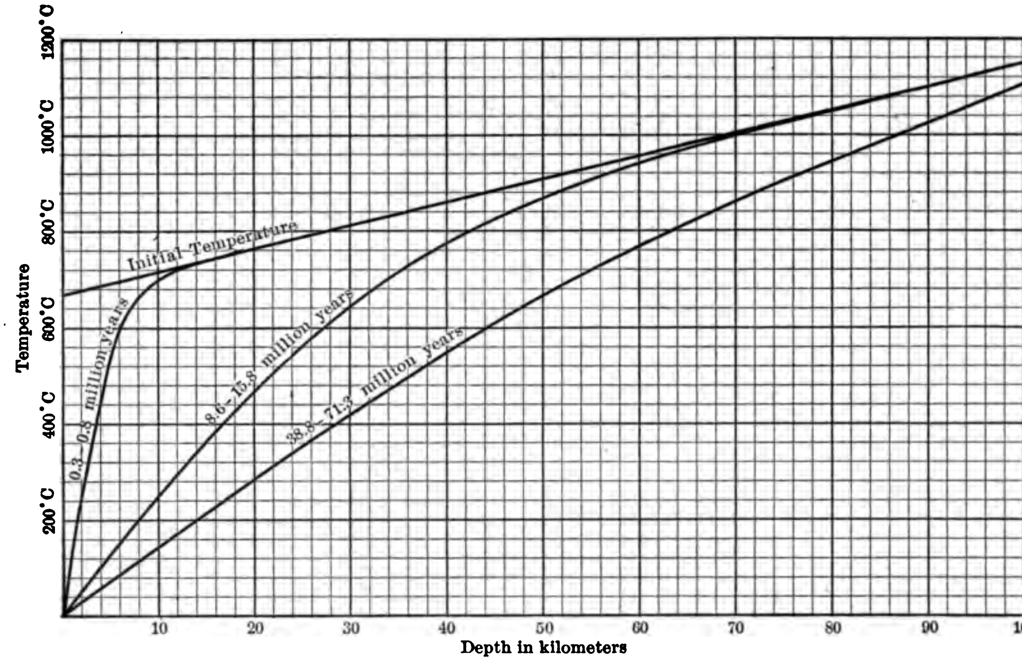


FIG. 14. Temperature curves for the earth, after cooling for the specified number of years, assuming the initial conditions of Art. 88

The smaller of the two times is for $h^2 = .0118$ (Kelvin), and the larger, for $h^2 = .0064$. It is to be noted that the temperature state at depths greater than 100 km. would be very little affected by cooling for even 50 million years

90. The Effect of Radioactivity on Earth Temperatures; Mathematical Treatment of a Special Case. While, as remarked above, we know too little of the actual conditions as regards the extent of distribution of radioactive substances in the earth to attempt any rigorous or complete treatment of their effect on the age and temperature of the earth, we can still solve the problem for specially assumed conditions. The assumptions we shall make are at least as consistent as any others with the facts as we now know them. The first is that only a fraction, $\frac{1}{n}$, of the total annual heat lost by the earth is due to radioactive causes. The rate of liberation of heat by the disintegration of such substances is supposed to be independent of the time, and the density of distribution of these heat-producing substances is assumed to fall off exponentially with increasing depth below the surface. It was mentioned above that some such assumption as this is practically necessary, for if these substances were scattered throughout the earth with their surface density of distribution, vastly more heat would be generated per year than is actually being conducted through the surface. The second assumption concerns the initial temperature state of the earth; that is, its temperature distribution at the time of the consistentior status. Instead of supposing, as in Kelvin's original calculation, that the earth was at a constant temperature at this time, we shall make the more reasonable assumption of Art. 88, which is based upon data obtained by Barus,* showing the relation of melting point to pressure to be nearly linear for a considerable depth.

In solving the problem we must first modify our fundamental conduction equation so as to take account of this continuous internal generation of heat. We found in Chapter II that the rate at which heat is added by conduction to any element of volume $dxdydz$ is

$$k\nabla^2\theta dxdydz.$$

* *Am. Journ. Sci.*, **45**, p. 16 (1895).

If in addition heat sources, such as these radioactive products, produce an amount of heat per second represented by

$$\phi(x, y, z, t)\, dxdydz,$$

then the temperature of this element will be raised at a rate $\frac{\partial\theta}{\partial t}$ such that

$$k\nabla^2\theta dxdydz + \phi(x, y, z, t)\, dxdydz = \frac{\partial\theta}{\partial t} cpdxdydz. \tag{63}$$

Therefore $$\frac{\partial\theta}{\partial t} = h^2\nabla^2\theta + \frac{\phi(x, y, z, t)}{cp}. \tag{64}$$

This is our fundamental equation. For linear flow it takes the form

$$\frac{\partial\theta}{\partial t} = h^2\frac{\partial^2\theta}{\partial x^2} + \frac{\phi(x, t)}{cp}. \tag{65}$$

In the present case the assumption is made that

$$\phi(x, t) = Ae^{-ax}, \tag{66}$$

where A is the quantity of heat generated per unit volume per second at the surface. Separate determinations of this quantity vary greatly, but the average result may be placed at about $.47 \times 10^{-12}$ calories per cubic centimeter per second for crustal rocks.* The total amount of heat generated in this way per second, and escaping through each square centimeter of the earth's surface, is

$$R = \int_0^\infty Ae^{-ax}dx = \frac{A}{a}. \tag{67}$$

But if T is the total amount of heat lost by the surface per square centimeter per second,

$$R = \frac{T}{n}. \tag{68}$$

When n is assumed, this enables us to determine a, since both A and T are known; that is, $a = \frac{nA}{T}$. (68 a)

Our fundamental equation (65) then becomes

$$\frac{\partial\theta}{\partial t} - h^2\frac{\partial^2\theta}{\partial x^2} = \frac{A}{cp}e^{-ax} = Be^{-ax}, \tag{69}$$

* From data furnished by Professor A. N. Winchell.

where B is written for $\frac{A}{c\rho}\cdot$ The solution of this equation must satisfy the boundary conditions

$$\theta = 0 \text{ at } x = 0, \tag{70}$$

$$\theta = mx + S \text{ when } t = 0. \tag{71}$$

We shall first change (69), by substitution, into a form which is homogeneous and linear. Assume that

$$\theta = u - \frac{B}{a^2h^2}e^{-ax}, \tag{72}$$

where u is some function of x and t. Then

$$\frac{\partial\theta}{\partial t} = \frac{\partial u}{\partial t}; \quad \frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{B}{h^2}e^{-ax}, \tag{73}$$

and (69) becomes,

$$\frac{\partial u}{\partial t} - h^2\frac{\partial^2 u}{\partial x^2} = 0. \tag{74}$$

The boundary conditions then become

$$u = \frac{B}{a^2h^2} \text{ at } x = 0, \tag{75}$$

$$u = mx + S + \frac{B}{a^2h^2}e^{-ax} \text{ when } t = 0. \tag{76}$$

As the problem would be much easier to handle if the first boundary condition were $u = 0$ at $x = 0$, we shall make the further substitution

$$v = u - \frac{B}{a^2h^2}, \tag{77}$$

which gives us, in place of (74),

$$\frac{\partial v}{\partial t} - h^2\frac{\partial^2 v}{\partial x^2} = 0; \tag{78}$$

and for boundary conditions

$$v = 0 \text{ at } x = 0,$$

$$v = f(x) = mx + \left(S - \frac{B}{a^2h^2}\right) + \frac{B}{a^2h^2}e^{-ax} \text{ when } t = 0. \tag{79}$$

This now becomes the problem of Art. 75, where was obtained the solution

$$v = \frac{1}{\sqrt{\pi}}\left\{\int_{\frac{-x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2} f(x + 2h\sqrt{t}\beta)\, d\beta \right.$$

$$\left. - \int_{\frac{+x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2} f(-x + 2h\sqrt{t}\beta)\, d\beta\right\}. \tag{80}$$

Substituting for $f(x + 2h\sqrt{t}\beta)$ and $f(-x + 2h\sqrt{t}\beta)$ from (79), this may be written

$$v = \frac{mx}{\sqrt{\pi}}\left[\int_{\frac{-x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2} d\beta + \int_{\frac{+x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2} d\beta\right]$$

$$+ \frac{S - \frac{B}{a^2h^2}}{\sqrt{\pi}}\left[\int_{\frac{-x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2} d\beta - \int_{\frac{+x}{2h\sqrt{t}}}^{\infty} e^{-\beta^2} d\beta\right]$$

$$+ \frac{2hm\sqrt{t}}{\sqrt{\pi}}\left[\int_{\frac{-x}{2h\sqrt{t}}}^{\infty} \beta e^{-\beta^2} d\beta - \int_{\frac{x}{2h\sqrt{t}}}^{\infty} \beta e^{-\beta^2} d\beta\right]$$

$$+ \frac{B}{a^2h^2\sqrt{\pi}}\left[e^{-ax}\int_{\frac{-x}{2h\sqrt{t}}}^{\infty} e^{-2ah\sqrt{t}\beta - \beta^2} d\beta - e^{ax}\int_{\frac{x}{2h\sqrt{t}}}^{\infty} e^{-2ah\sqrt{t}\beta - \beta^2} d\beta\right]. \tag{81}$$

Of the above four terms the first two can readily be shown to equal

$$mx \text{ and } \left(S - \frac{B}{a^2h^2}\right)\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2h\sqrt{t}}} e^{-\beta^2} d\beta \tag{82}$$

respectively, while the third vanishes. In evaluating the fourth we note that

$$\int e^{-(\beta^2 + 2ah\sqrt{t}\beta)}\, d\beta = e^{a^2h^2t}\int e^{-(\beta + ah\sqrt{t})^2}\, d\beta. \tag{83}$$

Making use of this fact, and of the substitution $\gamma = \beta + ah\sqrt{t}$, we have, finally, since $\frac{B}{a^2h^2} = \frac{A}{a^2k}$

and
$$\theta = v + \frac{A}{a^2k} - \frac{A}{a^2k}e^{-ax}, \tag{84}$$

$$\theta = mx + \left(S - \frac{A}{a^2k}\right)\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2h\sqrt{t}}} e^{-\beta^2} d\beta + \frac{A}{a^2k}\left[1 - e^{-ax} + \right.$$

$$\left.\frac{1}{\sqrt{\pi}}\left\{e^{a^2h^2t - ax}\int_{\frac{-x}{2h\sqrt{t}} + ah\sqrt{t}}^{\infty} e^{-\gamma^2} d\gamma - e^{a^2h^2t + ax}\int_{\frac{+x}{2h\sqrt{t}} + ah\sqrt{t}}^{\infty} e^{-\gamma^2} d\gamma\right\}\right]. \tag{85}$$

When $A = 0$, that is, when there is no radioactive material present, this solution reduces, as it should, to equation (60) of Art. 88.

A computation of the age of the earth has been made on the basis of (85) for the following assumed conditions: $A = .47 \times 10^{-12}$; $T = 1.285 \times 10^{-6}$; $n = 4$, that is, one fourth of the present heat loss is due to radioactivity; $k = .0045$; $c = .25$; $\rho = 2.8$; $m = .00005$; and $S = 995°$ C. Then the time required to cool from the initial conditions * of surface at 995° and temperature gradient of 5° per kilometer to the present surface gradient of 1° C. in 35 meters comes out to be 45.85×10^6 years. Without radioactivity the same initial conditions give 22.0×10^6 years, so we see that in this case the continuous generation of heat under these conditions increases the computed age of the earth by over 100 per cent.

It may be added that since the estimates of the earth's age based purely on refrigeration are of the same order of magnitude as those arrived at from geological considerations, such as stratigraphy, sodium denudation, etc., many geologists are inclined to believe that radioactivity is not as important in this connection as might be supposed; that is, that it contributes not more than about one fourth of the present total annual heat loss. If some such small fraction of the total heat loss is attributed to radioactive causes, estimates of the earth's age based on cooling will be in fair agreement with geological estimates.

PROBLEMS

1. Show that, under the conditions of Art. 75, if θ is initially equal to x, it will always be equal to x; and if it is initially x^2, its value at any time later will be given by

$$2\,hx\sqrt{\frac{t}{\pi}}\cdot e^{\frac{-x^2}{4h^2t}} + (2\,h^2t + x^2)\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2h\sqrt{t}}} e^{-\beta^2}d\beta.$$

* Strictly speaking, the conditions are really for a temperature of 1000° at a depth of 5 km. below the surface, the surface itself being, in accordance with the idea of the consistentior status, at or near zero in temperature. The above assumption of a surface initially at 995°, which is then suddenly cooled to and thereafter kept at zero, is made to render the problem mathematically simpler. That this would not substantially affect the result may be concluded from the curves of Fig. 14.

2. If the surface of dry soil ($h^2 = .0031$), initially at 2° C. throughout, is lowered to − 30° C., how long before the zero temperature will penetrate to the depth of 10 cm.? 1 m.? (Compare Problem 7, p. 75.) (77 min.; 5.3 days.)

3. An enormous mass of cast iron at 100° C., with one plane face, is dropped into water at 10° C. Assuming no convection currents in the water (these would be minimized by choosing the face horizontal and on the underside), what will be the temperature of the surface of contact? How long before a point 2 m. inside the surface will fall in temperature to 95° C.? (90.3° C.; 4.5 days.)

4. In the preceding problem calculate at what rate heat is passing out through each square meter of the boundary surface after 10 min. (694 cal./sec.)

5. Show by a method of reasoning similar to that of Art. 75, that if the plane surface of the solid is made impervious to heat, instead of being kept at constant temperature, then

$$\theta = \frac{1}{2h\sqrt{\pi t}}\int_0^\infty f(\lambda)\left\{e^{\frac{-(\lambda-x)^2}{4h^2t}} + e^{\frac{-(\lambda+x)^2}{4h^2t}}\right\}d\lambda.$$

6. Water pipes are buried 1 m. below the surface in concrete masonry ($h^2 = .0058$), the whole being at 8° C. If the surface temperature is lowered to − 20° C., how long before the pipes are in danger of freezing? (9 days.)

7. If the initial temperature of the earth was 3900° C. throughout and it has been cooling 100 million years since then, with the surface at zero, plot its present state of temperatures as a function of the distance below the surface. (Use Kelvin's constants; that is, $h^2 = .0118$ and $k = .0042$.)

8. Under the conditions of the previous problem compute the present loss of heat per square centimeter of surface per year. How thick a layer of ice would this melt? (49 cal.; 0.6 cm.)

CASE III

Heat Sources

91. We shall now make use of the conception of a *heat source*, an idea which has been used very successfully by Lord Kelvin * and other writers in handling problems in heat flow. If a certain amount of heat is suddenly developed in each unit of area of a plane surface in a body, this surface becomes an *instantaneous source* of heat, while if the heat is developed continuously instead of suddenly, it is known as a *permanent source*.†

* *Math. and Phys. Papers*, II, pp. 41 ff.

† The problem of Art. 90 involved a special case of permanent sources with a volume distribution.

92. Let q units of heat be suddenly generated on each unit area of a plane in an infinite body, or on each unit area in some cross section of a long rod whose surface is impervious to heat. If the material is of density ρ and specific heat c, the unit of heat will raise unit volume of the material $\frac{1}{\rho c}$ degrees. The quantity

$$Q \equiv \frac{q}{\rho c} \tag{86}$$

is called the *strength* of this instantaneous source. If q units are produced in each second, then Q' is the strength of the permanent source.

93. Regard the plane $x = \lambda$ over which the instantaneous source of heat is spread as of thickness $\Delta\lambda$; then its temperature when the heat is suddenly generated will be raised by

$$\frac{q}{\rho c \Delta\lambda} = \frac{Q}{\Delta\lambda} \text{ degrees,} \tag{87}$$

and we have a case to be handled by (20). The temperature at any point will be given by

$$\theta = \frac{Q}{2\, h \Delta\lambda \sqrt{\pi t}} \int_{\lambda}^{\lambda + \Delta\lambda} e^{\frac{-(\lambda - x)^2}{4 h^2 t}} \cdot d\lambda, \tag{88}$$

since $f(\lambda) = 0$ outside these limits of integration. Now let the mean value of $e^{\frac{-(\lambda - x)^2}{4 h^2 t}}$ between the above limits be $e^{\frac{-(\lambda' - x)^2}{4 h^2 t}}$ where $\lambda < \lambda' < (\lambda + \Delta\lambda)$. Then

$$\theta = \frac{1}{2\, h \sqrt{\pi t}} \cdot \frac{Q}{\Delta\lambda} e^{\frac{-(\lambda' - x)^2}{4 h^2 t}} \cdot \Delta\lambda, \tag{89}$$

which, as $\Delta\lambda \doteq 0$, approaches the limit

$$\theta = \frac{Q}{2\, h \sqrt{\pi t}} e^{\frac{-(\lambda - x)^2}{4 h^2 t}}, \tag{90}$$

where the heat source is at a plane λ distant from the origin. Shifting this to the origin, (90) becomes

$$\theta = \frac{Q}{2\, h \sqrt{\pi t}} e^{\frac{-x^2}{4 h^2 t}}. \tag{91}$$

94. This gives us temperatures at any point for any time if we have a linear flow of heat from an instantaneous source of strength Q at the origin, the temperature of all other parts being initially zero. It is well to test the correctness of this solution by seeing if we can derive from it what is an inevitable conclusion from the conditions given, namely, that the total amount in the material at any time is just equal to the original amount q (per unit area of section). From (91) the quantity of heat in any element dx is

$$\theta\rho c dx = \frac{q}{2h\sqrt{\pi t}} e^{\frac{-x^2}{4h^2t}} \cdot dx; \tag{92}$$

whence the total amount present in the body at any time is represented by

$$\int_{-\infty}^{+\infty} \theta\rho c dx = \frac{q}{2h\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{\frac{-x^2}{4h^2t}} \cdot dx \tag{93}$$

$$= \frac{q}{2h\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\gamma^2} \cdot 2h\sqrt{t} d\gamma \tag{94}$$

$$= q. \tag{95}$$

Since the additive effect of any number of such sources could be obtained by a summation of such terms as (90), formula (20) may be regarded as applying to the case in which we start with an instantaneous source of strength $f(\lambda)d\lambda$ in each element of length, $d\lambda$, of the solid or bar in the x-direction.

95. Since it appears on expanding (91) in a series that $\theta = 0$ $(x \neq 0)$ when $t = 0$ and also when $t = \infty$, it must have a maximum value at some time t_1. To get this, differentiate (91) and equate to zero,

$$\frac{\partial \theta}{\partial t} = \frac{Q}{2h\sqrt{\pi}} \left\{ \frac{1}{\sqrt{t}} \frac{\partial}{\partial t} e^{\frac{-x^2}{4h^2t}} + e^{\frac{-x^2}{4h^2t}} \left(-\frac{1}{2} t^{-\frac{3}{2}} \right) \right\} = 0, \tag{96}$$

from which

$$t_1 = \frac{x^2}{2h^2}. \tag{97}$$

Putting this value of t in (91), we get for the value of this maximum

$$\theta_1 = \frac{Q}{x\sqrt{2\pi e}}. \tag{98}$$

The maximum temperatures, then, for various points are inversely proportional to the distances of these points from the instantaneous source, while these maxima are reached in times directly proportional to the squares of the distances.

APPLICATIONS

96. Electric Welding. Two round iron bars 8 cm. (3.1″) in diameter are being electrically welded end to end. If a current of 30,000 amperes at 4 volts is required for 4 sec., and if this energy is supposed to be all developed at the plane of contact, how far from the end will the temperature of 1200° C. (2192° F.) penetrate, supposing the initial temperature of the bars to be 0°C.?

The total heat developed will be

$$30{,}000 \times 4 \times 4 \text{ joules} = \frac{480000}{4.2} \text{ cal., or } \frac{480000}{4.2 \times 16\,\pi} \text{ cal./sq. cm.};$$

that is, $Q = 2740.$*

Hence we have, from (98),

$$1200 = \frac{Q}{x\sqrt{2\pi e}} = \frac{2740}{x \cdot 4.13},$$

or $x = .55$ cm.; that is, the temperature of 1200° C. will penetrate to a depth not greater than .55 cm. (.22″) — somewhat less, in fact, since the generation of heat is not instantaneous, as the solution assumes.

97. Casting. A large plate of iron 3 cm. (1.2″) thick is to be cast in a sand mold. Assuming that the pouring temperature is 1620° C. (2948° F.) while the mold is at 20° C. (68° F.), what will be the maximum temperature in the mold 10 cm. (4″) from the plate, and when will this occur?

We shall neglect the thickness of the plate, that is, consider it a plane source, and also neglect the changes in the thermal constants of iron with high temperatures. Then

$$q = 3 \times 7.85 \times .1055 \times 1600 \text{ cal./cm}^2.,$$

or $$Q = 4800.$$

* Strictly speaking, the time element in the generation of this heat must be overlooked in this solution.

So from (98), with a 20° shift of our temperature scale, and with $h^2 = .0049$, that is, using the same diffusivity as for soil,

$$\theta_1 = \frac{4800}{10 \cdot 4.13} = 116^\circ \text{ C.},$$

which will occur, from (97), at

$$t = 10{,}200 \text{ sec.}$$

For half the distance away this temperature would be 232° C. and the corresponding time a quarter as large as before. Adding 20° C. to shift our scale back again, these temperatures become, at 10 cm. (4″), 136° C. (277° F.); and at 5 cm. (2″), 252° C. (485° F.).

The solution of the first of the above problems gives an idea of how far from the welded joint we might expect to find the grain of the material injured by overheating, while from the second we could draw some conclusion as to how near such a casting, wood, say, might be safely located in the mold.

98. Temperatures in Decomposing Granite. We shall now take up a problem involving permanent sources with a volume distribution. While of some interest from the geological standpoint, it is difficult, and the solution of only one or two particular cases will be attempted.

It has been noted in some instances that areas of granite undergoing decomposition are several degrees warmer than the surrounding rock. It is known that granite gives out heat during decomposition, the total amount being of the order of 100 cal. per gram, but it is an extremely slow process, and our problem is to see if any reasonable assumption of the rate at which such heat is given off would serve to explain this increased temperature.

99. To be able to treat the case as a specific problem we shall assume first that the decomposing granite is in the form of a wall of thickness l, whose faces are kept at zero. Then if A' calories are generated per second per cubic centimeter of the decomposing material, we have for our fundamental equation

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2} + \frac{A'}{c\rho} \qquad (99)$$

with boundary conditions

$$\theta = 0 \text{ at } x = 0 \text{ and } x = l \tag{100}$$

and
$$\theta = 0 \text{ when } t = 0. \tag{101}$$

Let
$$u = \theta + \phi(x) \tag{102}$$

where $\phi(x)$ is a function of x (only), yet to be determined. Replacing θ by u in (99),

$$\frac{\partial u}{\partial t} - h^2\left(\frac{\partial^2 u}{\partial x^2} - \phi''(x)\right) = \frac{A'}{c\rho} \equiv A. \tag{103}$$

But if we determine $\phi(x)$ so that

$$\phi''(x) = \frac{A}{h^2}, \tag{104}$$

or
$$\phi(x) = \frac{Ax^2}{2h^2} + bx + d, \tag{105}$$

then
$$\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}.$$

To satisfy (100) and also make $u = 0$ at $x = 0$ and $x = l$, $\phi(x)$ must vanish at $x = 0$ and $x = l$; therefore

$$d = 0, \text{ and } b = -\frac{Al}{2h^2}. \tag{106}$$

So then
$$\phi(x) = \frac{A}{2h^2}(x^2 - lx) \tag{107}$$

and
$$u = \theta + \frac{A}{2h^2}(x^2 - lx), \tag{108}$$

or
$$\theta = u + \frac{A}{2h^2}(xl - x^2). \tag{109}$$

The solution of the problem is then merely a question of determining u under the following conditions:

Fundamental equation,
$$\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}. \tag{110}$$

Boundary conditions,
$$u = 0 \text{ at } x = 0 \text{ and } x = l, \tag{111}$$

$$u = f(x) = \frac{A}{2h^2}(x^2 - lx) \text{ when } t = 0. \tag{112}$$

This is nothing but the problem of the slab with faces at zero, which will be treated in Case IV, next to be considered. While in this particular example the form of $f(x)$ makes the determination of u a rather lengthy process, it offers no special difficulties and gives us as a final solution of the problem

$$\theta = \frac{A'}{2k}\left\{xl - x^2 - \frac{8l^2}{\pi^3}\sum_{m=2p+1}^{m=\infty}\frac{1}{m^3}e^{-\frac{h^2m^2\pi^2t}{l^2}}\cdot\sin\frac{m\pi x}{l}\right\}. \quad (113)$$

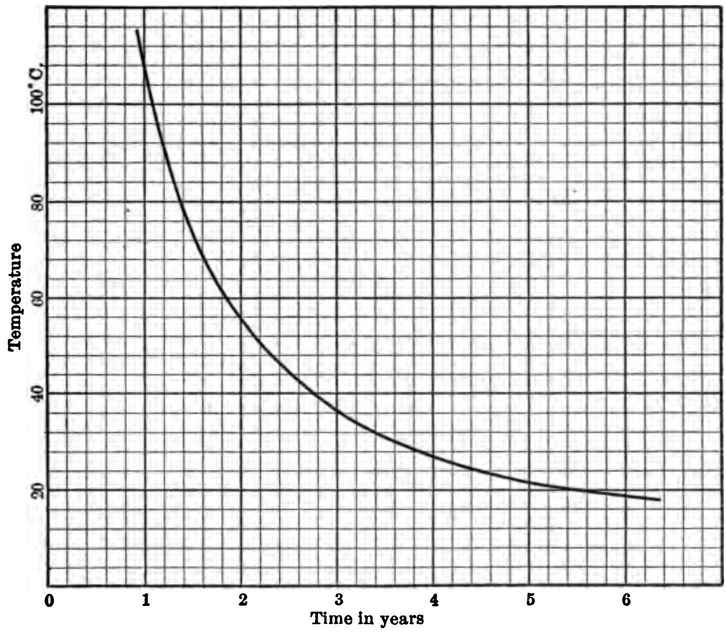


FIG. 15. Curve showing the relation between the final temperature in the center of a granite layer or wall 915 cm. (30′) thick and the total time necessary to effect its decomposition, computed for the conditions of Art. 99

The curve of Fig. 15 has been computed with the use of the equation above, the rate A' of heat generation being chosen so that the entire process of decomposition with the resultant generation of 100 cal. per gram takes place in n years. The thickness of granite is taken as 915 cm. (30′), and the time chosen as that for the completion of the process. The diffusivity is taken as .0155.

100. A second hypothetical case, much simpler than the above, is as follows: Suppose that this wall or slab of decomposing granite l cm. thick is in contact on each side with ordinary granite. Suppose also that this slab is initially heated to some temperature θ_0 about 50° C. above that of the surrounding rock and allowed to cool for a year. This gives a temperature at the center, as may be readily computed from (25), of $.355\,\theta_0$, or about 17.7° above that of the surrounding rock at some distance away. Now by differentiation of (25) with respect to x and multiplication by .0081, the conductivity of granite, we get the rate of heat flow out through each face of this slab as

$$\frac{k\theta_0}{2\,h\sqrt{\pi t}}\left(1-e^{-\frac{l^2}{4h^2t}}\right)=.000057 \text{ cal. per sq. cm. per second,}$$

for $l = 915$ cm.

So far we have taken no account of the heat of decomposition, for the above discussion is merely to find a reasonable assumption for the temperature distribution in this slab and the surrounding rock as we find it at present. We may now question at what rate decomposition would have to take place in order to furnish heat at just the rate required to maintain this temperature state steady for some time, and at once compute this rate as such that the 100 cal. would be liberated, that is, the process finished, in about sixty-eight years.

The preceding discussion should enable the geologist to form some idea of the temperatures which might be caused by or explained by decomposition. As the rate of such decomposition is generally supposed to be very much slower than that taken above, it is evident that a large thickness of such decomposing granite would be required to cause even a few degrees of excess temperature.

PROBLEMS

1. Derive (20) and (29) on the basis of heat sources (see Art. 94).

2. In electrically welding two large copper bars 2640 cal. are suddenly developed in each square centimeter of the contact plane. Assuming the initial temperature to be 20° C., when will the maximum occur at 40 cm. from this plane and what will be its value? (706 sec.; 39.7° C.)

3. A plate of lead 1 cm. thick is cast in a sand mold ($h^2 = .0049$). If the mold is initially at zero while the lead is poured at 450° C., what will be the maximum temperature at 3 cm. away, and when will this occur?

(36.3° C.; 918 sec.)

4. Show from (90) that if we have a permanent source of constant strength Q' located in a plane distant λ from the origin, which begins to liberate heat in a body initially at zero at time $t = 0$, then the temperature at any time t will be given by

$$\theta = \frac{Q'}{2h\sqrt{\pi}}\int_0^t e^{\frac{-(\lambda - x)^2}{4h^2(t-\tau)}} \cdot (t-\tau)^{-\frac{1}{2}} d\tau.$$

Case IV

Solid with Two Parallel Bounding Planes — the Slab

101. In this case we have to deal with a body bounded by two parallel planes distant l apart, with the initial temperature condition of the body given. The problem is to find the subsequent temperature for any point. The solution will of course fit equally well the case of a short rod with protected surface.

102. Both Faces at Zero. The boundary conditions here are

$$\theta = 0 \text{ at } x = 0, \qquad (114)$$

$$\theta = 0 \text{ at } x = l, \qquad (115)$$

$$\theta = f(x) \text{ when } t = 0. \qquad (116)$$

Now we have already seen (Art. 66) that

$$\theta = e^{-h^2\gamma^2 t} \cdot \sin \gamma x \qquad (117)$$

and

$$\theta = e^{-h^2\gamma^2 t} \cdot \cos \gamma x \qquad (118)$$

are particular solutions of the fundamental equation

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}. \qquad (1)$$

Form (117) satisfies (114) for any value of γ, and also (115) if $\gamma = \frac{m\pi}{l}$ where m is a whole number. It does not, as it stands, fulfill (116), but it may be possible to combine a number of

terms like (117) and secure an expression which shall be a solution of (1) and which satisfies (116). For

$$\theta = A_1 e^{\frac{-h^2\pi^2 t}{l^2}} \cdot \sin\frac{\pi x}{l} + A_2 e^{\frac{-4h^2\pi^2 t}{l^2}} \cdot \sin\frac{2\pi x}{l} + A_3 e^{\frac{-9h^2\pi^2 t}{l^2}} \cdot \sin\frac{3\pi x}{l} + \cdots \tag{119}$$

is still a solution of (1), satisfying (114) and (115), which reduces, when $t=0$, to

$$\theta = A_1 \sin\frac{\pi x}{l} + A_2 \sin\frac{2\pi x}{l} + A_3 \sin\frac{3\pi x}{l} + \cdots; \tag{120}$$

and from Art. 58 this equals $f(x)$ if the function fulfills the conditions of Art. 51 between 0 and l, and if

$$A_m = \frac{2}{l}\int_0^l f(\lambda)\sin\frac{m\pi\lambda}{l}\,d\lambda. \tag{121}$$

The solution of our problem then is

$$\theta = \frac{2}{l}\sum_{m=1}^{m=\infty}\left\{e^{\frac{-h^2m^2\pi^2 t}{l^2}} \cdot \sin\frac{m\pi x}{l}\int_0^l f(\lambda)\sin\frac{m\pi\lambda}{l}\,d\lambda\right\}. \tag{122}$$

103. Adiabatic Cases — Slab with Nonconducting Faces. If the faces instead of being kept at constant temperature are impervious to heat, we shall have the same differential equation but quite different boundary conditions; namely,

$$\frac{\partial\theta}{\partial x} = 0 \text{ at } x = 0, \tag{123}$$

$$\frac{\partial\theta}{\partial x} = 0 \text{ at } x = l, \tag{124}$$

$$\theta = f(x) \text{ when } t = 0. \tag{125}$$

Conditions (123) and (124) are fulfilled by solution (118) if $\gamma = \frac{m\pi}{l}$ just as before, and (125) may be satisfied by combining a number of terms of this type. This gives

$$\theta = \frac{2}{l}\left\{\frac{1}{2}\int_0^l f(\lambda)\,d\lambda + \sum_{m=1}^{m=\infty}\left(e^{\frac{-h^2m^2\pi^2 t}{l^2}} \cdot \cos\frac{m\pi x}{l}\int_0^l f(\lambda)\cos\frac{m\pi\lambda}{l}\,d\lambda\right)\right\}. \tag{126}$$

104. If only one face is nonconducting, the other being kept at zero, the solution is contained in equation (122). This may be shown by the same considerations which were used in Art. 70, that is, by imagining a nonconducting plane cutting through the center of a slab of double thickness, parallel to its faces, where the temperature conditions are supposed perfectly symmetrical on each side of such a plane. There would then be no tendency to a flow of heat across such a surface, and hence placing a nonconducting division plane there and removing half of the slab will not affect the solution in any way. Therefore in handling a problem of this nature, that is, one face impervious to heat, we solve it as a case of a slab of *twice* the thickness, and the temperatures of the nonconducting face would be found as those at the *middle* of the slab of double thickness.

APPLICATIONS

105. The Theory of the Fireproof Wall. With the aid of the foregoing deductions we can now develop a theory which finds immediate application to a large number of practical problems, namely, that of heat penetration into a slab or wall, one side of which is subjected to sudden heating, as by fire; or, as we shall call it for brevity, the "theory of the fireproof wall." It is to be understood that this theory applies only to the purely thermal aspects of the question of fire-protecting walls and floors, and not at all to the very important considerations of strength, ability to withstand heating and quenching, and other questions which must be largely determined by experiment.

We shall treat the problem for four cases of somewhat differing conditions. It is assumed in all cases that the wall is relatively homogeneous in structure, a condition which would be fulfilled by practically all masonry or concrete walls, floors, or chimneys. For hollow tiling or other cellular structure the theory would not apply directly, but would still afford at least an indication of the laws for these cases. It is also assumed that the wall is initially at about the same temperature throughout its thickness, as would be true in almost every practical example.

106. Case A. The conditions assumed for this case are that the front face of the wall is suddenly raised to the temperature θ_0 and maintained there, while the rear face is protected so that it suffers no loss of heat. It is desired to know the rise in temperature of the rear face for various intervals of time. The latter condition is fulfilled sufficiently well by a wall which is backed by wood, that is, door casing, or better by a concrete or masonry floor on which is piled poorly conducting (for example, combustible) material.

As explained in Art. 104, such a case as this, involving an impervious surface, can be treated as that of a slab of twice the thickness, the rear (impervious) face of the wall corresponding to the middle of the slab ($x = \frac{1}{2}l$). To apply (122) we must proceed as in Art. 77, writing $-\theta_0$ for $f(\lambda)$. Then

$$\theta_1 = \frac{-2\,\theta_0}{l}\sum_{m=1}^{m=\infty}\left\{e^{\frac{-h^2m^2\pi^2t}{l^2}}\cdot\sin\frac{m\pi x}{l}\int_0^l\sin\frac{m\pi\lambda}{l}\,d\lambda\right\} \tag{127}$$

This corresponds to a slab initially at $-\theta_0$, while the faces are at zero. Adding θ_0 to each side, we then have as the solution for the present case

$$\theta = \theta_0\left\{1 - \frac{2}{l}\sum_{m=1}^{m=\infty}e^{\frac{-h^2m^2\pi^2t}{l^2}}\cdot\sin\frac{m\pi x}{l}\int_0^l\sin\frac{m\pi\lambda}{l}\,d\lambda\right\}, \tag{128}$$

where l is *twice* the thickness of the wall. For the rear face the sine term becomes $\sin\frac{1}{2}m\pi$, so that only terms in odd values of m are present. Hence we may write (128) after evaluating the integral

$$\theta = \theta_0\left\{1 - \frac{4}{\pi}e^{-\frac{h^2\pi^2t}{l^2}} + \frac{4}{3\pi}e^{-\frac{9h^2\pi^2t}{l^2}} - \frac{4}{5\pi}e^{-\frac{25h^2\pi^2t}{l^2}} + \cdots\right\}. \tag{129}$$

In most cases the terms after the third in the above series are negligible.

107. Case B. This differs from the preceding in that the temperature of the front face is supposed to rise gradually instead of suddenly. If the rise is rapid at first, as it would be in most cases, — for example, if the wall were exposed to a flame, — an approximate solution may be arrived at by the device suggested

in discussing the removal of shrunk-on fittings (Art. 84); that is, the assumption of an added thickness whose outer surface is suddenly raised to, and kept at, a constant temperature θ_0'. By properly choosing θ_0', as well as the thickness to be added, a temperature-time curve can be found for the plane representing the original surface, nearly like many actual heating curves; the computation is then carried out accordingly. The results obtained, however, are generally only slightly different from those for Case A if the mean value of θ_0 is used.

108. Case C. We have here an important difference to take account of in the conditions. While the front surface is supposed to be suddenly brought to the temperature θ_0, as in Case A, the rear surface in the present case is supposed to lose heat by radiation and convection instead of being protected, and hence will not rise to as high a temperature as in Case A.

The rigorous handling of this problem is extremely difficult and would be well beyond the limits of the present work, but, as in many previous cases, it is still possible to reach a solution accurate enough for all practical purposes, and at not too great an expense of labor. This may be done as follows: In the treatment of the semi-infinite solid with boundary at zero (Art. 75) we found that the equations could be deduced from those for the infinite solid by a suitable assumption for the temperatures on the negative side of the origin, that is, for $f(-\lambda)$, the latter being so determined that the boundary should remain constantly at zero. Now if the boundary instead of being at zero radiates with an emissivity E, this condition can be introduced * by putting into the relation (identical with (20))

$$\theta = \frac{1}{2h\sqrt{\pi t}}\int_0^\infty \left(f(\lambda)\, e^{\frac{-(\lambda - x)^2}{4h^2t}} + f(-\lambda)\, e^{\frac{-(\lambda + x)^2}{4h^2t}} \right) d\lambda \quad (130)$$

the condition that

$$f(-\lambda) = f(\lambda) - 2\frac{E}{k} e^{-\frac{E}{k}\lambda} \cdot \int_0^\lambda f(\gamma)\, e^{\frac{E}{k}\gamma}\, d\gamma. \quad (131)$$

* See Weber-Riemann, *Part. Diff. Gleichungen*, II, Art. 39.

This gives the temperatures for a semi-infinite medium with radiating surface and initial temperature conditions determined by $f(\lambda)$. Now let us make the assumption that $f(\lambda)$ has the value zero for a distance b from the radiating face, and $2\theta_0$ from there to infinity. This gives the somewhat complicated equation

$$\theta = \frac{2\theta_0}{\sqrt{\pi}} \int_{\frac{b-x}{2h\sqrt{t}}}^{\frac{b+x}{2h\sqrt{t}}} e^{-\beta^2}\, d\beta$$

$$+ 2\theta_0\, e^{(b+x)\frac{E}{k} + \frac{E^2}{k^2}h^2 t} \cdot \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{b+x+2h^2\frac{E}{k}t}{2h\sqrt{t}}} e^{-\beta^2}\, d\beta\right), \quad (132)$$

and if we investigate with the aid of this equation the temperature in the plane distant b from the radiating face, we find that, for small values of E and not too small values of b, this is almost constant for a considerable time and has the value θ_0.

We have, then, the solution of our problem in the above equation. This plane which is kept at θ_0 corresponds to the front face of the wall whose thickness is b, and the temperatures of the rear or radiating face will be given by putting $x = 0$ in this equation. The value of the emissivity constant E may be taken for small ranges of temperature at about .0003 calories per second per square centimeter per degree centigrade above the temperature of the surroundings, for an average surface such as a wall (see Appendix A). Strong convection such as a wind, or higher temperature differences, will increase this figure considerably; in some cases, however, it may be even less than the above value.

To gain some idea of the difference of the results for this case and for Case A, a few computations have been carried out with (132) and plotted in Fig. 16. These are for a wall of stone concrete ($h^2 = .0058$) 20.3 cm. (8″) thick, whose front face is heated to θ_0. For two hours, under these conditions, the temperatures of the rear face for Case C are lower than they would be for Case A in the ratio of 35 to 53. For a granite wall ($h^2 = .0155$) with the same thickness and same time this ratio would be about 28 to 35.

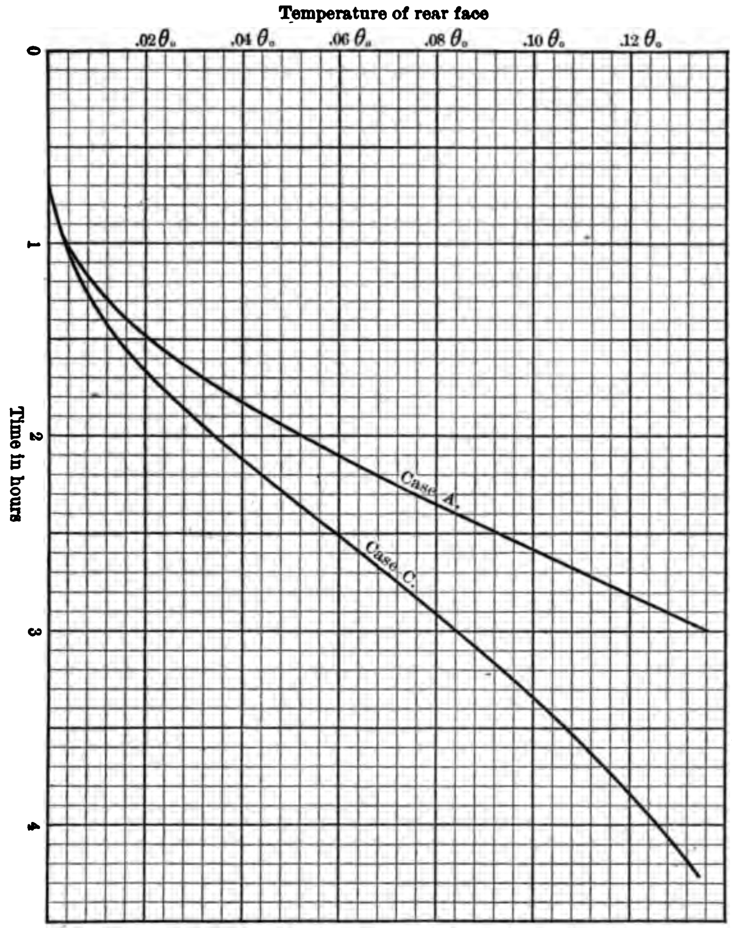


FIG. 16. Temperatures of the rear face of a concrete wall 20.3 cm. (8″) thick, whose front face is heated to θ_0; computed for the conditions of Cases A and C

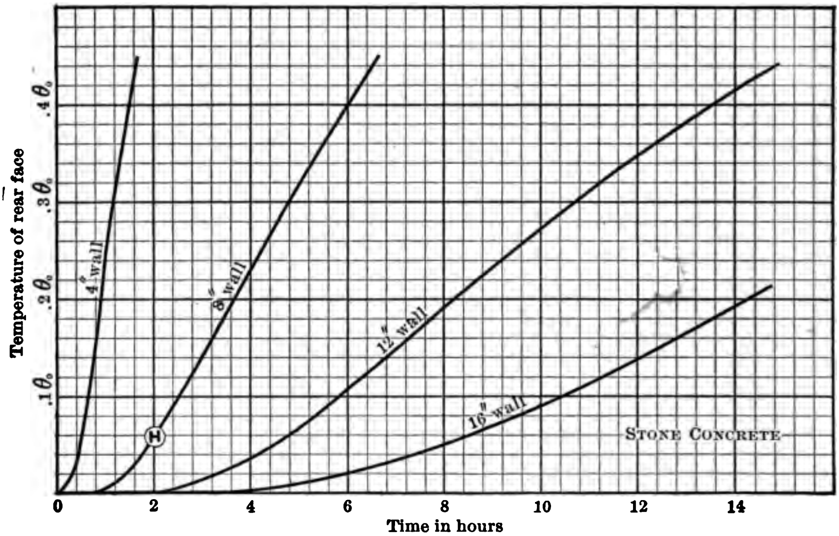


FIG. 17. Computed curves showing the rise in temperature of the rear faces of walls of stone concrete ($h^2 = .0058$), whose front faces are suddenly heated to, and afterward maintained at, θ_0. See Arts. 111 and 112

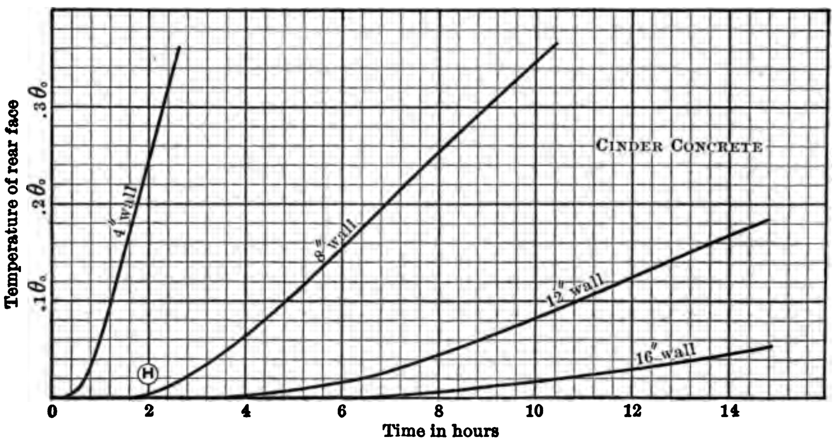


FIG. 18. Computed temperature-time curves for the rear faces of walls of cinder concrete ($h^2 = .0031$)

109. Case D. This differs from the last only in the supposition that the temperature rises gradually instead of suddenly. No attempt * will be made at treating this case mathematically, but from the conclusions reached for Case B we are reasonably safe in handling it as Case C, using a mean value for the temperature θ_0.

110. Discussion of the General Principles. Having treated in detail the several cases, we may now draw some general conclusions in regard to thermal insulation under fire conditions. From the preceding discussion we see that Case A is the one from which we can most safely make these deductions; for B and D are more or less minor modifications, while C would invariably lead to lower results. Hence for a margin of safety we shall make our deductions largely from (the ideal) Case A.

The first conclusion to be drawn from (129) is that the temperature of the rear face is a function of h^2 rather than of k. In other words, the insulating value of material for such a wall is dependent not alone on its conductivity, but rather on its conductivity divided by the product of its specific heat and density; that is, its *diffusivity*. Material for such purpose should therefore have as low a conductivity and as high a density and specific heat as possible, for if the density happens to be low, it may prove no better insulator than something of higher conductivity but of correspondingly higher density. This is true in some cases for slag or cinder concrete,† although the latter has as a rule a lower diffusivity than stone concrete.

The second conclusion from (129) is that any change which alters t and l^2 in the same proportion does not affect the temperature θ of the rear surface of the wall. In other words, for a given temperature rise of the rear face the time will vary as the *square* of the thickness. Since one measure of the effectiveness of such a fireproof wall or floor would be the time to which it

* For a fairly approximate treatment the method used for Case B might be followed; that is, the assumption of a small added thickness.

† See Woolson, *Proc. Am. Soc. for Testing Materials*, **6**, p. 438 (1906). He notes that in one case cinder concrete gave hardly better insulation tests than gravel concrete. The explanation is evident from the above discussion.

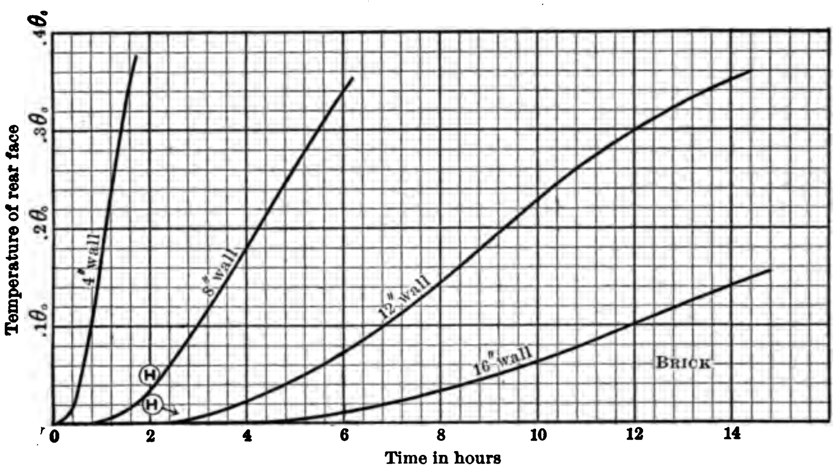


FIG. 19. Computed temperature-time curves for the rear faces of walls of building brick ($h^2 = .0050$)

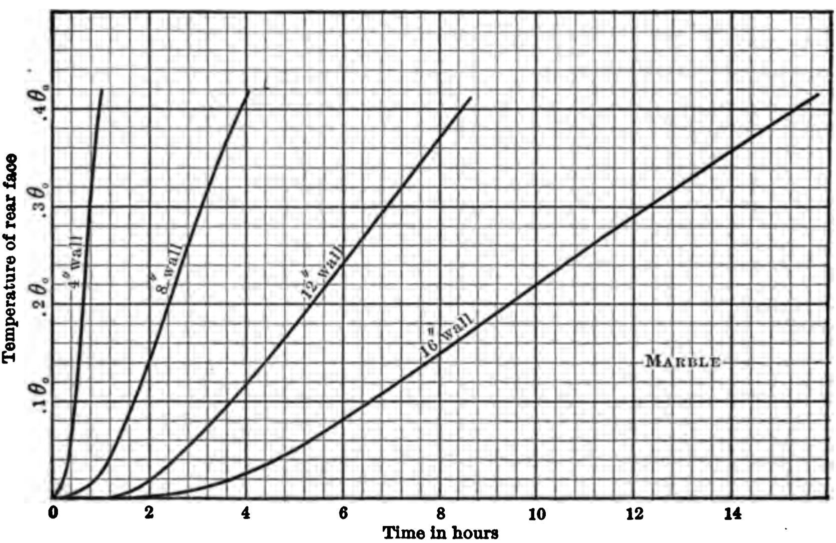


FIG. 20. Computed temperature-time curves for the rear faces of walls of marble ($h^2 = .0090$)

would delay the penetration of a dangerously high temperature to the rear face, this makes the efficiency of such wall or floor proportional to the square of its thickness (compare the "law of times" in Art. 78).

111. These conclusions are represented graphically in the curves of Figs. 17–21. The temperature θ of the rear face of a wall whose front face is at θ_0 is expressed for various times

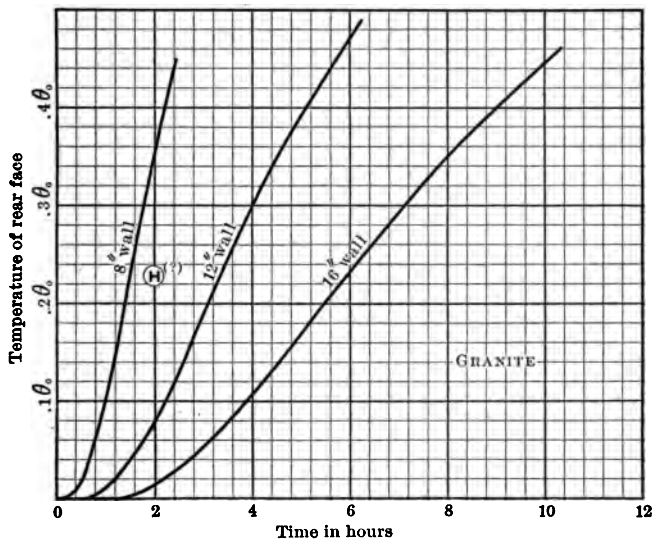


FIG. 21. Computed temperature-time curves for the rear faces of walls of granite ($h^2 = .0155$)

and thicknesses of wall in fractions of θ_0. For convenience in the computation of all such cases the following table has been prepared. In this, x represents the quantity

$$.4343\frac{h^2\pi^2 t}{l^2},$$

where l is double the thickness of the wall, while y is the temperature of the rear face expressed as a fraction of θ_0. It will of course serve equally well to determine the temperature for the middle point of a slab of thickness l, whose initial temperature is zero and whose faces are kept at θ_0.

TABLE

Values of the function $y = 1 - \frac{4}{\pi}\left\{10^{-x} - \frac{1}{3}10^{-9x} + \frac{1}{5}10^{-25x} - \frac{1}{7}10^{-49x} + \dots\right\}$

x	y	x	y	x	y	x	y
.01	.0000	.11	.0546	.25	.2864	.80	.7982
.02	.0000	.115	.0619	.26	.3022	.85	.8202
.03	.0000	.12	.0692	.27	.3178	.90	.8397
.035	.0001	.125	.0768	.28	.3331	1.00	.8727
.04	.0005	.13	.0848	.29	.3480	1.10	.8989
.045	.0010	.135	.0927	.30	.3727	1.25	.9284
.05	.0021	.14	.1009	.32	.3912	1.50	.9597
.055	.0037	.145	.1092	.34	.4184	1.75	.9774
.06	.0055	.15	.1176	.36	.4444	2.00	.9873
.065	.0081	.16	.1345	.38	.4693	2.25	.9928
.07	.0113	.17	.1517	.40	.4931	2.50	.9960
.075	.0150	.18	.1690	.45	.5482	2.75	.9977
.08	.0194	.19	.1862	.50	.5974	3.00	.9987
.085	.0241	.20	.2033	.55	.6412	3.25	.9993
.09	.0294	.21	.2204	.60	.6802	3.50	.9996
.095	.0351	.22	.2372	.65	.7150	3.75	.9998
.10	.0412	.23	.2539	.70	.7460	4.00	.9999
.105	.0478	.24	.2702	.75	.7736		

112. Experimental. To secure a reasonable check on the preceding conclusions the following simple experiment was tried by the authors: A plate of hard unglazed porcelain .905 cm. thick was heated on one surface by the sudden application of hot mercury, and the temperature rise of the other surface, which was protected from loss of heat by loose cotton wrappings, was measured with a small thermo-element. The process was then repeated for a similar plate of thickness 1.780 cm., the temperatures being plotted in Fig. 22. As the diffusivity of the porcelain was not known, it was computed from the determination for the thinner plate that $\theta = \frac{1}{2}\theta_0$ at time 52 sec. This gives $h^2 = .0060$, and the two theoretical curves were calculated from this value. Two plates of each thickness were tested, and it is to be noted that the agreement with the theoretical curve is at least as close as

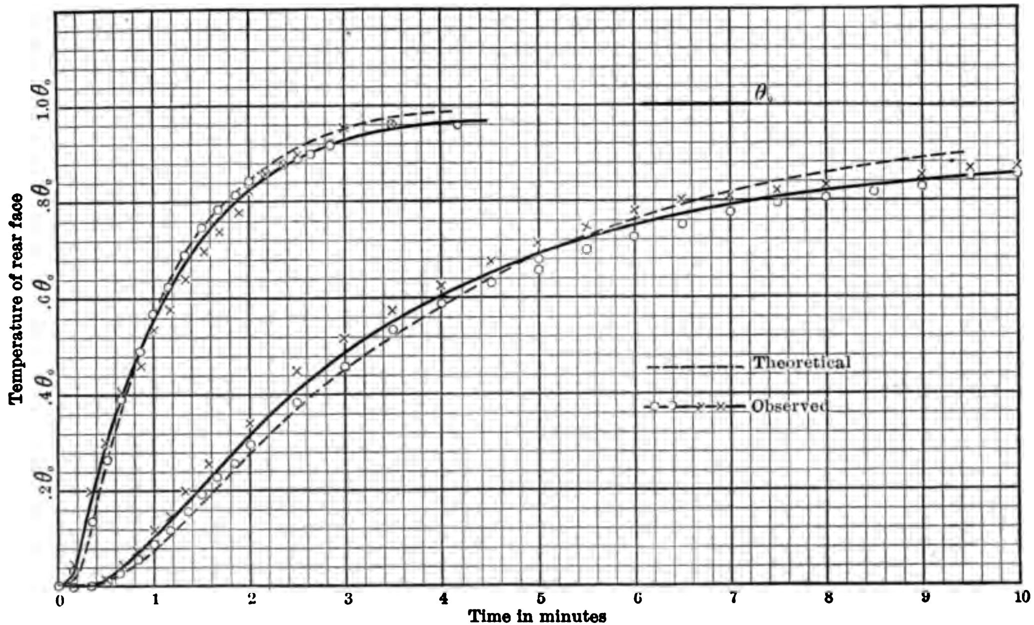


FIG. 22. Theoretical and observed temperature-time curves for the rear faces of miniature walls of porcelain ($h^2 = .0060$), the temperature of the front faces being suddenly raised to θ_0 and maintained there during the experiment

that between the two sets of observations. This is a very satisfactory proof of the "law of times."

On a larger scale there are available the fire tests on various walls, made by R. L. Humphrey.* These were two-hour tests, mostly on 8″ walls, the temperature θ_0 of the front faces being in the neighborhood of 700° C. His results have been plotted, where possible, in the curves of Figs. 17–21, being denoted by the symbol ⊖. The agreement (overlooking radiation losses) for the case of stone concrete is very good, his point representing the mean of three concordant results. As the diffusivity (.0058) used in computing the curves of Fig. 17 was obtained from an entirely different source,† this may be regarded as a reasonably good check on the theory. The other agreements are poor, but no worse than might be expected, considering the varying qualities of such materials and our imperfect knowledge of their thermal constants. In general it appears that cracks and fissures which develop in the heated surface under such tests cause at first a more rapid penetration of heat than the theory would anticipate.

The effect of steel reënforcement in such a wall is of interest in this connection. A consideration of (129) shows that a steel wall should afford the same temperature rise of the rear face under the same heating conditions of the front face as a concrete wall of about one fifth the thickness. Placing a steel bar running through the wall, therefore, would be almost equivalent, thermally, to drilling a hole of the same diameter four fifths through the wall.

113. Copper Converter. We may make brief mention of a number of other problems to which the foregoing principles apply directly. For example, take the case of the copper converter, a somewhat shallow container lined with magnesia fire brick in many cases about 30.5 cm. (1′) thick, in which molten copper at an average temperature of perhaps 1300° C. is kept for two

* United States Geological Survey, *Bull. No. 370.* The authors are also indebted to Mr. Rudolph P. Miller, Superintendent of Buildings, New York City, for much valuable information and data on tests.

† Norton, *Proc. Nat. Assoc. Cement Users*, 7, p. 78.

or three hours: how hot may the outside of the brick be expected to get if the radiation from the surface is small?

Using $h^2 = .0074$ and $l = 61$, we find, with the use of the Table of Art. 111, that the temperature of the outside would rise only 8° C. in 2 hr., while in 4 hr. it should not exceed 95° C. The conductivity and diffusivity of magnesia brick are sometimes quoted as half as large again as the values here used. In this latter case the temperatures would be 42° and 230°.

114. Fire Brick. In a number of practical cases it is desirable to know to what extent and also how rapidly the temperature in the inside of a brick follows that of the outside. This is of particular interest in connection with the burning of brick, and also in the case of the "regenerator," where heat from flue gases is stored up in a checkerwork wall of fire brick, to be utilized shortly in heating other gases. Using $h^2 = .0074$, we find that the center of such a brick 6.35 cm. (2.5″) thick — the larger dimensions being of little influence if the two flat sides are exposed — will rise in 5 min. to .26 of the temperature of the faces, in 10 min. to .57, and in 20 min. to .85. For building brick of perhaps two thirds this diffusivity the figures would be .12 for 5 min., .38 for 10 min., and .70 for 20 min.

115. Optical Mirrors. In the process of finishing huge telescopic mirrors it is necessary that they be allowed to remain in a constant-temperature room before testing, until the glass is at sensibly the same temperature throughout. For such a glass mirror ($h^2 = .0057$) 25 cm. thick, x (in the Table of Art. 111) is $.0000395\,t$. Then if the surface temperature is changed by θ_0 and we wish to be sure that the change at the center is, say, 90% of this, we find for $y = .9$ that $x = 1.10$. Therefore $t = 27{,}700$ sec., or the mirror should be kept at this constant temperature θ_0 for at least 7.7 hr. For 14.2 hr. the figure would be 98.7%.

The above is on the assumption that both faces of the glass plate are exposed. If only one face is exposed, it would be the protected face rather than the center which would be the last to assume the constant temperature, and the delay would be four times as long.

116. Annealing Castings. While a large number of other applications of the foregoing theory might be mentioned, such as numerous cases of fireplace insulation, resister-furnace insulation, fireproof-safe construction, and the like, we shall content ourselves with one more example, namely, the problem of annealing steel castings; that is, the question of how long the heating must continue to bring the interior temperature to the desired value. This is quite similar in principle to the illustration of the last article, and we may readily compute that for a mild steel casting ($h^2 = .173$) in the form of a plate 30.5 cm. (1′) in thickness, it would take 23 min. for the center to rise to within 90% of the temperature of the faces, providing these were quickly raised to their final temperature. For a plate of half this thickness it would take only one quarter the time, etc.

If the faces are gradually rather than suddenly heated, the process would take longer, but would have the advantage that in this case the difference in temperature between outside and inside would be lessened, as well as the time it would be necessary to keep the faces at the highest temperature.

PROBLEMS

1. A plate of steel ($h^2 = .121$) of thickness 2.54 cm. and temperature 0° C. is to be tempered by immersion in a bath of stirred molten metal at $\theta°$. How long should it be left to assure that the steel is throughout within 98% of this higher temperature? (23 sec.)

2. A fireplace is insulated from wood by 15 cm. of fire brick. If the face is kept for some time at 400° C., how long before the wood at the rear will char, supposing this to occur at 250° C.? How long for a thickness of 25 cm.? (4.2 hr.; 11.6 hr.)

3. Compare the results for the three following problems based on Cases I, II, and IV of the present chapter. A plate of copper 10 cm. thick and at θ_0 is placed between two large slabs of similar material at zero: how long before the center will fall in temperature to $\frac{1}{2}\theta_0$? If instead of a plate we have a large mass originally at θ_0, while the surface is afterwards kept at zero, how long before the temperature 5 cm. in from the surface will fall to $\frac{1}{2}\theta_0$? If the slab is of the same thickness as in the first case, but the faces are kept at zero, solve this same problem for the center.

(24.4 sec.; 24.4 sec.; 8.3 sec.)

4. A sheet of ice 5 cm. thick, in which the temperature varies uniformly from zero on one face to $-20°$ on the other, has its faces protected by an impervious covering. What will be the temperature of each face after 10 min.? $(-10.56°$ and $-9.44°.)$

Case V

Long Rod with Radiating Surface

117. This differs from Cases I and II of this chapter in that there is a continual loss of heat by radiation from the surface of the rod. We have already handled the steady state for this case in Arts. 16–21, where we found that the Fourier equation had to be modified by the addition of a term taking account of the radiation, and became

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2} - b^2 \theta. \tag{133}$$

We shall assume as before that the rod is so thin that the temperature is sensibly uniform over the cross section, and that the surroundings are at zero.

118. Initial Heat Distribution given. We must seek a solution of (133), subject to the conditions

$$\theta = f(x) \text{ when } t = 0, \tag{134}$$

$$\theta = 0 \text{ when } t = \infty. \tag{135}$$

Now the substitution $$ue^{-b^2 t} = \theta \tag{136}$$

reduces (133) at once to $$\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}, \tag{137}$$

where u fulfills the condition

$$u = f(x) \text{ when } t = 0, \tag{138}$$

and, indirectly (135), since u is finite. But this is identical with Case I, so the solution for u is given by equation (22) of Art. 67. Using this, we may write at once

$$\theta = \frac{e^{-b^2 t}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(x + 2h\sqrt{t}\beta) e^{-\beta^2} d\beta. \tag{139}$$

In other words, this differs from the nonradiating case only by the factor $e^{-b^2 t}$.

119. One End of Rod at Zero; Initial Distribution of Heat given. The boundary conditions are

$$\theta = 0 \text{ at } x = 0, \tag{140}$$

$$\theta = f(x) \text{ when } t = 0. \tag{141}$$

If we make the substitution (136), then u must satisfy (137) and also the conditions

$$u = 0 \text{ at } x = 0, \tag{142}$$

$$u = f(x) \text{ when } t = 0. \tag{143}$$

As this is the case already treated in Art. 75, we may write, using (31),

$$\theta = \frac{e^{-b^2t}}{\sqrt{\pi}}\left\{\int_{\frac{-x}{2h\sqrt{t}}}^{x} f(2\,\beta h\sqrt{t}+x)e^{-\beta^2}\,d\beta \right.$$
$$\left. -\int_{\frac{+x}{2h\sqrt{t}}}^{\infty} f(2\,\beta h\sqrt{t}-x)e^{-\beta^2}\,d\beta\right\}. \tag{144}$$

120. End of Rod at Constant Temperature θ_0; Initial Temperature of Rod Zero. We cannot solve this problem directly, like the two preceding, as an extension of cases already worked out; for the boundary condition $\theta = \theta_0$ at $x = 0$ would mean $u = \theta_0 \cdot e^{-b^2t}$ at $x = 0$, which would not fit any case we have treated. But we can handle this case with equation (144) by the aid of an ingenious device * whereby we first solve the problem for the boundary conditions

$$\theta = 0 \text{ at } x = 0, \tag{145}$$

$$\theta = -\,\theta_0 e^{\frac{-bx}{h}} \text{ when } t = 0. \tag{146}$$

Applying (144) to this case, we get, on simplifying,

$$\theta = \frac{\theta_0}{\sqrt{\pi}}\left\{e^{\frac{bx}{h}}\int_{\frac{x}{2h\sqrt{t}}}^{\infty} e^{-(b\sqrt{t}+\beta)^2}\,d\beta - e^{\frac{-bx}{h}}\int_{\frac{-x}{2h\sqrt{t}}}^{\infty} e^{-(b\sqrt{t}+\beta)^2}\,d\beta\right\}. \tag{147}$$

Now

$$\theta = \theta_0 e^{\frac{-bx}{h}} \tag{148}$$

* Cf. Byerly, *Fourier's Series and Integrals*, p. 90.

is a particular solution of (133), as is also (147) just preceding, so the sum of (147) and (148),

$$\theta = \frac{\theta_0}{\sqrt{\pi}}\left\{\sqrt{\pi}e^{\frac{-bx}{h}} + e^{\frac{bx}{h}}\int_{\frac{x}{2h\sqrt{t}}}^{\infty} e^{-(b\sqrt{t}+\beta)^2}d\beta \right.$$
$$\left. - e^{\frac{-bx}{h}}\int_{\frac{-x}{2h\sqrt{t}}}^{\infty} e^{-(b\sqrt{t}+\beta)^2}d\beta\right\}, \qquad (149)$$

is still a solution of (133), which, moreover, fits our present boundary conditions, namely,

$$\theta = \theta_0 \text{ at } x = 0, \qquad (150)$$

$$\theta = 0 \text{ when } t = 0. \qquad (151)$$

We may simplify it somewhat by writing

$$\gamma = b\sqrt{t} + \beta, \qquad (152)$$

and hence $$d\gamma = d\beta, \qquad (153)$$
in (149). This gives

$$\theta = \theta_0\left\{e^{\frac{-bx}{h}} + \frac{e^{\frac{bx}{h}}}{\sqrt{\pi}}\int_{b\sqrt{t}+\frac{x}{2h\sqrt{t}}}^{\infty} e^{-\gamma^2}d\gamma - \frac{e^{\frac{-bx}{h}}}{\sqrt{\pi}}\int_{b\sqrt{t}-\frac{x}{2h\sqrt{t}}}^{\infty} e^{-\gamma^2}d\gamma\right\}. \qquad (154)$$

121. A careful examination of this expression is worth while to be sure that it is the desired solution. For $t = 0$ and $x \neq 0$ the lower limit of the first integral becomes ∞, hence the integral vanishes; while in the second integral it becomes $-\infty$, giving a value of $\sqrt{\pi}$ to the integral. Hence for $t = 0$ we have $\theta = 0$, as it should be for all cross sections of the rod except the heated end. As both integrals have the same limiting value for $x \doteq 0$, this gives the right temperature for the end, namely, $\theta = \theta_0$. Both integrals vanish for $t = \infty$, and so, for the steady state, we have the result deduced in Arts. 16 and 18,

$$\theta = \theta_0 e^{\frac{-bx}{h}}. \qquad (155)$$

From the value for b^2 given in (17), Art. 17, namely, $\frac{h^2Ep}{kS}$, we see that b^2 is very small if the emissivity is very small.

Setting $b^2 = 0$ in (154), we get

$$\theta = \theta_0 \left\{ 1 + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{2h\sqrt{t}}}^{\infty} e^{-\gamma^2} d\gamma - \frac{1}{\sqrt{\pi}} \int_{\frac{-x}{2h\sqrt{t}}}^{\infty} e^{-\gamma^2} d\gamma \right\}, \quad (156)$$

which is readily seen to be identical with the results of Art. 77 for the linear flow of heat in an infinite body.

PROBLEM

1. A wrought-iron rod 1 cm. in diameter and 1 m. long is shielded with an impervious covering and subjected to temperatures 0° C. and 100° C. at its ends, until a steady state is reached. The covering is then removed and the rod placed in close contact at its ends with two long similar rods at zero, the temperature of the air being zero also. If E is .0003, what will be the temperature at the middle of the meter rod after 15 min.? (Compare Problem 6, p. 75.) (13.5°.)

CHAPTER VIII

THE FLOW OF HEAT IN MORE THAN ONE DIMENSION

122. In this chapter we shall consider a few of the many heat-conduction problems involving more than one dimension. In particular we shall take up the case of the radial flow of heat, including "cooling of the sphere" problems; also the general case of three-dimensional conduction.

CASE I

Radial Flow. Initial Temperature given as a Function of the Distance from a Fixed Point

123. This is the case analogous to the first discussed under linear flow in the last chapter, the essential difference being that the isothermal surfaces instead of being plane are here spherical. In the discussion of the steady state for radial flow (Art. 30), we had occasion to express Fourier's equation in terms of the variable r, finding that

$$\nabla^2\theta = \frac{1}{r}\frac{\partial^2(r\theta)}{\partial r^2}, \tag{1}$$

the partial notation being used here to show differentiation with respect to r alone, θ now depending on t as well; so the fundamental equation becomes

$$\frac{\partial\theta}{\partial t} = \frac{h^2}{r}\frac{\partial^2(r\theta)}{\partial r^2}, \tag{2}$$

or

$$\frac{\partial(r\theta)}{\partial t} = h^2\frac{\partial^2(r\theta)}{\partial r^2}. \tag{3}$$

The solution of our problem must satisfy this equation, and the boundary condition

$$\theta = f(r) \text{ when } t = 0. \tag{4}$$

Let

$$u = r\theta, \tag{5}$$

and our differential equation (3) reduces to

$$\frac{\partial u}{\partial t} = h^2\frac{\partial^2 u}{\partial r^2}, \tag{6}$$

where $$u = rf(r) \text{ when } t = 0 \tag{7}$$

and $$u = 0 \text{ at } r = 0, \tag{8}$$

u being always positive if θ is taken as positive. But the solution of (6) under these conditions will be identical with that for the case of linear flow with one face at zero, treated in Art. 75. Using, as in this case, λ as the variable of integration, and remembering that when $t = 0$

$$u = \lambda f(\lambda), \tag{9}$$

we have the temperature at any distance r from the point, given, from (29) of Art. 75, by the equation

$$u = r\theta = \frac{1}{2h\sqrt{\pi t}}\left\{\int_0^\infty \lambda f(\lambda) e^{\frac{-(\lambda - r)^2}{4h^2t}} d\lambda - \int_0^\infty \lambda f(\lambda) e^{\frac{-(\lambda + r)^2}{4h^2t}} d\lambda\right\}. \tag{10}$$

With the substitutions

$$\beta = \frac{\lambda - r}{2h\sqrt{t}}, \text{ or } \lambda = r + 2h\sqrt{t}\beta,$$

and $$\beta' = \frac{\lambda + r}{2h\sqrt{t}}, \text{ or } \lambda = -r + 2h\sqrt{t}\beta', \tag{11}$$

this becomes

$$\theta = \frac{1}{r\sqrt{\pi}}\left\{\int_{\frac{-r}{2h\sqrt{t}}}^\infty (r + 2h\sqrt{t}\beta) f(r + 2h\sqrt{t}\beta) e^{-\beta^2} d\beta - \int_{\frac{+r}{2h\sqrt{t}}}^\infty (-r + 2h\sqrt{t}\beta') f(-r + 2h\sqrt{t}\beta') e^{-\beta'^2} d\beta'\right\}. \tag{12}$$

124. If the initial temperature is a constant, θ_0, within a sphere in the solid of radius R, and zero everywhere outside, the subsequent temperatures are given from (10) by

$$\theta = \frac{\theta_0}{2rh\sqrt{\pi t}}\left\{\int_0^R \lambda e^{\frac{-(\lambda - r)^2}{4h^2t}} d\lambda - \int_0^R \lambda e^{\frac{-(\lambda + r)^2}{4h^2t}} d\lambda\right\}; \tag{13}$$

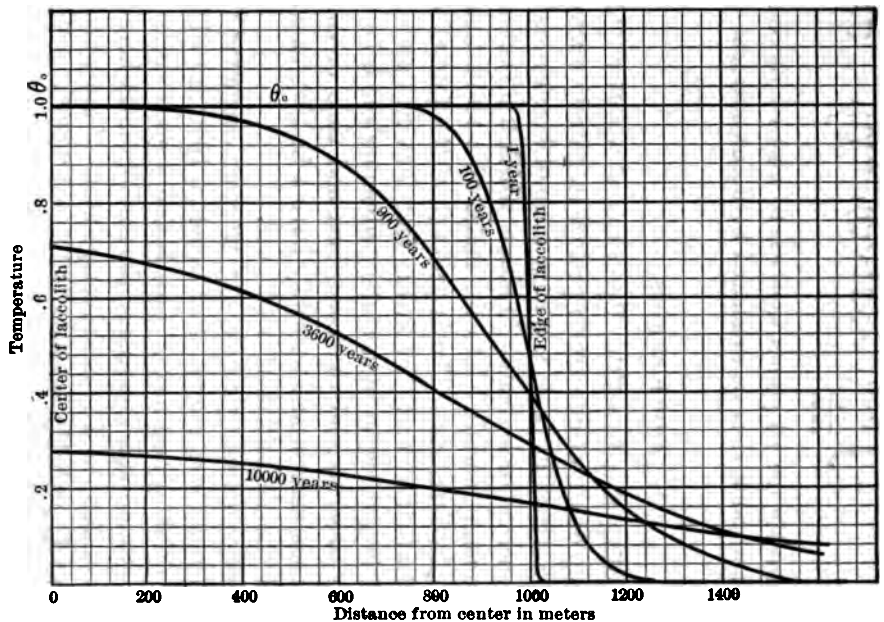


FIG. 23. Computed temperature curves for a laccolith 1000 meters in radius, which has been cooling from an initial temperature θ_0, for various periods of time

A point 5 meters from the boundary surface would reach its maximum temperature in about 100 years, while at 100 meters the maximum would not be reached for over 1000 years

or, from (12), by

$$\theta = \frac{\theta_0}{r\sqrt{\pi}}\left\{\int_{\frac{-r}{2h\sqrt{t}}}^{\frac{R-r}{2h\sqrt{t}}}(r+2\,h\sqrt{t}\beta)\,e^{-\beta^2}d\beta \right.$$

$$\left. - \int_{\frac{+r}{2h\sqrt{t}}}^{\frac{R+r}{2h\sqrt{t}}}(-r+2\,h\sqrt{t}\beta)\,e^{-\beta^2}d\beta\right\}. \qquad (14)$$

This gives θ directly for all points save $r = 0$, where it becomes indeterminate and must then be evaluated by differentiation.

APPLICATIONS

125. The Cooling of a Laccolith. By means of equation (14) we can solve a problem of interest to geologists, namely, that of the cooling of a laccolith. This is a huge mass of igneous rock, more or less spherical or lenticular in shape, which has been intruded in a molten condition into the midst of a sedimentary rock, for example, limestone. The importance of the formation, from a geological standpoint, lies in the fact that ores are frequently found in the region immediately adjoining the original surface of the laccolith, and the conditions and time of cooling of the igneous mass would naturally have a bearing on any explanation of the deposition of such ores.

The temperature curves given in Fig. 23 were computed for the following conditions: radius, R, of laccolith, 1000 m.; diffusivity = .0118 (Kelvin's estimate. This is also a good mean of the value for granite and limestone; the medium must here be assumed to be uniform). The initial temperature of the igneous rock is taken as θ_0, probably between 1000° and 2000° C., while the surrounding rock is assumed at zero.

The conclusions to be drawn from the curves are: first, that the cooling is a very slow process, occupying tens of thousands of years; second, that the boundary-surface temperature quickly falls to half * the initial value and then cools only slowly, and

* Strictly speaking, the initial temperature of the boundary surface would be somewhat higher than this; for the conductivity of hot igneous rock is considerably

also that for a hundred or more years there is a large temperature gradient over only a few meters and a very slow progress of the heat wave; third, the maximum temperature in the limestone, or the crest (so to speak) of the heat wave, travels outward only a few centimeters a year. The mass behind it will then suffer a contraction as soon as it begins to cool, and the cracking and introduction of mineral-bearing material* is doubtless a consequence of this.

PROBLEM

1. Molten copper at 1085° C. is suddenly poured into a spherical cavity in a large mass of copper at 0° C. If the radius of the cavity is 20 cm., find the temperature at a point 10 cm. from the center after 15 min. Neglect the latent heat of fusion. (24°.)

Case II

Instantaneous Heat Source at a Point

126. If q units of heat are suddenly developed at a point in the interior of a solid which is everywhere else at zero, a radial flow will at once take place and the temperature at any point for any subsequent time can be found in terms of the time and the distance from this center. This case is analogous to that discussed in Art. 93, where we had a linear flow from an instantaneous heat source located in a plane of infinitesimal thickness. Just as in this case, too, we can deduce the solution by a special application of a more general one. For if in (13) we let the radius R of the spherical region, which is initially at constant temperature θ_0, become vanishingly small, while its initial temperature is correspondingly increased so as to make the amount of heat finite, we shall have a solution of the present problem.

To get this, put

$$q = \theta_0 c\rho \cdot \tfrac{4}{3} \pi R^3 \tag{15}$$

greater than that of the cold limestone, although, in order to be able to handle the problem, we have had to consider their thermal constants as the same. The temperature of the boundary surface for the first hundred years or so could best be estimated from equation (49) of Art. 80. The error introduced by assuming the diffusivities to be the same becomes less and less as the cooling proceeds.

* See Leith and Harder, United States Geological Survey, *Bull. No. 338.*

as the amount of heat in a very small sphere of radius R, and substitute the value of θ_0 deduced from this in (13). Then

$$\theta = \frac{3q}{4\,c\rho R^3\pi^{\frac{3}{2}}\cdot 2\,rh\sqrt{t}}\left\{\int_0^R \lambda e^{\frac{-(\lambda-r)^2}{4h^2t}}d\lambda - \int_0^R \lambda e^{\frac{-(\lambda+r)^2}{4h^2t}}d\lambda\right\}. \quad (16)$$

Now we may write

$$e^{\frac{-(\lambda-r)^2}{4h^2t}} = e^{\frac{-\lambda^2}{4h^2t}}\cdot e^{\frac{2\lambda r}{4h^2t}}\cdot e^{\frac{-r^2}{4h^2t}} \quad (17)$$

$$= \left(1 - \frac{\lambda^2}{4\,h^2t} + \frac{\lambda^4}{\underline{|2}\,(4\,h^2t)^2} - \cdots\right)$$

$$\cdot\left(1 + \frac{2\,\lambda r}{4\,h^2t} + \frac{1}{\underline{|2}}\frac{4\,\lambda^2r^2}{(4\,h^2t)^2} + \cdots\right)e^{\frac{-r^2}{4h^2t}}, \quad (18)$$

since
$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \cdots. \quad (19)$$

We can see by inspection the similar expression for $e^{\frac{-(\lambda+r)^2}{4h^2t}}$. As λ is a very small quantity in this integration, being confined to the limits 0 and R, (18) simplifies to

$$\left(1 + \frac{2\,\lambda r}{4\,h^2t}\right)e^{\frac{-r^2}{4h^2t}}, \quad (20)$$

the effect of the other terms vanishing in the limit as $R \doteq 0$, as may be readily seen on inspection of (21).

Then (16) becomes

$$\theta = \frac{3q}{4\,c\rho R^3\pi^{\frac{3}{2}}\cdot 2\,rh\sqrt{t}}\cdot e^{\frac{-r^2}{4h^2t}}\left\{\int_0^R \lambda\left(1 + \frac{2\,\lambda r}{4\,h^2t}\right)d\lambda\right.$$

$$\left. - \int_0^R \lambda\left(1 - \frac{2\,\lambda r}{4\,h^2t}\right)d\lambda\right\} \quad (21)$$

$$= \frac{3q}{4\,c\rho R^3\pi^{\frac{3}{2}}\cdot 2\,rh\sqrt{t}}\cdot e^{\frac{-r^2}{4h^2t}}\cdot\frac{R^3r}{3\,h^2t} \quad (22)$$

$$= q\,\frac{h^2}{k}\left(\frac{1}{2\,h\sqrt{\pi t}}\right)^3\cdot e^{\frac{-r^2}{4h^2t}}, \quad (23)$$

since
$$c\rho = \frac{k}{h^2}.$$

127. Having derived (23) it may be instructive to reverse the process and show that it is our desired solution. To do this we must show that it satisfies (3) and the boundary conditions

$$\theta = 0 \text{ when } t = \infty, \tag{24}$$

$$\theta = 0 \text{ when } t = 0, \text{ save at } r = 0, \tag{25}$$

and also the condition that the total amount of heat at any time shall equal q.

Differentiation gives

$$\frac{\partial (r\theta)}{\partial t} = \left(-\frac{3}{2t} + \frac{r^2}{4h^2t^2}\right) r\theta, \tag{26}$$

$$\frac{\partial (r\theta)}{\partial r} = \left(\frac{1}{r} - \frac{r}{2h^2t}\right) r\theta, \tag{26 a}$$

$$\frac{\partial^2 (r\theta)}{\partial r^2} = \left(-\frac{3}{2h^2t} + \frac{r^2}{4h^4t^2}\right) r\theta, \tag{26 b}$$

showing that (3) is satisfied. That conditions (24) and (25) are fulfilled may be shown if we rewrite that part of (23) containing t,

$$\frac{1}{t^{\frac{3}{2}}e^{\frac{a}{t}}} = \frac{1}{(t^{\frac{3}{2}})\left(1 + \frac{a}{t} + \frac{a^2}{\lfloor 2\, t^2} + \cdots\right)}. \tag{27}$$

The denominator is seen to be infinite for $t = 0$ or ∞; hence (23) vanishes for each of these values. As to the last condition, the total amount of heat is given by

$$\int_0^\infty \rho c\theta 4\pi r^2 dr = \int_0^\infty q\left(\frac{1}{2h\sqrt{\pi t}}\right)^3 e^{\frac{-r^2}{4h^2t}} \cdot 4\pi r^2 dr. \tag{28}$$

If we put

$$\gamma = \frac{r}{2h\sqrt{t}}, \tag{29}$$

the second member becomes

$$\frac{4q}{\sqrt{\pi}} \int_0^\infty e^{-\gamma^2} \cdot \gamma^2 d\gamma, \tag{30}$$

which (Appendix C) is equal to q.

128. The time t_1 at which θ reaches its maximum value is given by differentiating (23) and equating to zero. This gives

$$t_1 = \frac{r^2}{6\,h^2}. \tag{31}$$

The corresponding value of the temperature is

$$\theta_1 = \left(\frac{1}{\sqrt{\frac{2}{3}\pi e}}\right)^3 \cdot \frac{q}{c\rho r^3}. \tag{32}$$

129. Equation (23) shows that θ has a value different from zero in all parts of space even when t is exceedingly small, or, in other words, that heat is propagated apparently with an infinite velocity. As a matter of fact the heat disturbance is undoubtedly transmitted with great rapidity through the medium, although it is continually losing so much energy to this medium, which it has to heat up as it passes through it, that the actual amount of heat which travels to any appreciable distance from the source in a very short time is very small.

PROBLEMS

1. A 50-g. bullet is cast in a wrought-iron mold. Assuming the pouring temperature at 450° C. and the mold at zero, and neglecting the heat of fusion of lead, find the temperature 3 cm. away from the bullet in 10 sec.; also 6 cm. away. (Neglect dimensions of bullet.) (2.23°; .046°.)

2. If heat equivalent to the combustion of 1,000,000 kg. of anthracite (heat of combustion 8000 cal/g.) is suddenly generated at a point in the earth, when will the maximum temperature occur at a point 50 m. distant and what will be its value? (Use $h^2 = .0064$, $k = .0045$.) (20 years; 6° C.)

Case III

The Cooling of a Sphere with Surface at Constant Temperature

130. Surface at Zero. To solve this problem we must find a solution of (3), which satisfies the boundary conditions,

$$\theta = f(r) \text{ when } t = 0, \tag{33}$$

$$\theta = 0 \text{ at } r = R. \tag{34}$$

Making the substitution $\qquad u = r\theta, \qquad (35)$

(3) reduces to
$$\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial r^2}, \tag{36}$$

where u must fulfill the conditions

$$u = rf(r) \text{ when } t = 0, \tag{37}$$

$$u = 0 \text{ at } r = R, \tag{38}$$

$$u = 0 \text{ at } r = 0. \tag{39}$$

It will be seen that this makes the problem similar to that of the slab (Art. 102) with faces at temperature zero, and initial temperature $rf(r)$. With the aid of (122) we may then write

$$\theta = \frac{u}{r} = \frac{2}{Rr}\sum_{m=1}^{m=\infty} \sin\frac{m\pi r}{R} \cdot e^{\frac{-h^2m^2\pi^2t}{R^2}} \int_0^R \lambda f(\lambda) \sin\frac{m\pi\lambda}{R}\,d\lambda. \tag{40}$$

If the initial temperature is a constant, θ_0, we may write (40)

$$\theta = \frac{2\,\theta_0}{Rr}\sum_{m=1}^{m=\infty} \sin\frac{m\pi r}{R} \cdot e^{\frac{-h^2m^2\pi^2t}{R^2}} \cdot \int_0^R \lambda \sin\frac{m\pi\lambda}{R}\,d\lambda. \tag{41}$$

But
$$\int_0^R \lambda \sin\frac{m\pi\lambda}{R}\,d\lambda = -\frac{R^2}{m\pi}\cos m\pi, \tag{42}$$

so that (40) may be written for this case

$$\theta = \frac{2\,R\theta_0}{\pi r}\left\{\sin\frac{\pi r}{R} \cdot e^{\frac{-\pi^2h^2t}{R^2}} - \frac{1}{2}\sin\frac{2\,\pi r}{R} \cdot e^{\frac{-4\pi^2h^2t}{R^2}} + \frac{1}{3}\sin\frac{3\,\pi r}{R} \cdot e^{\frac{-9\pi^2h^2t}{R^2}} - \cdots\right\}. \tag{43}$$

131. Initial Temperature Zero; Surface at θ_0. By the method of Art. 77 we may at once write this equation for the case of the initial temperature zero and surface at θ_0 as

$$\theta = \theta_0\left(1 - \frac{2\,R}{\pi r}\left\{\sin\frac{\pi r}{R} \cdot e^{\frac{-\pi^2h^2t}{R^2}} - \frac{1}{2}\sin\frac{2\,\pi r}{R} \cdot e^{\frac{-4\pi^2h^2t}{R^2}} + \cdots\right\}\right), \tag{44}$$

while by a suitable shift of the temperature scale it may of course be applied to the case of any two constant temperatures, initial and surface.

132. The average temperature θ' of the sphere for any time t may be found from (43) by multiplying each element of volume by its corresponding temperature, summing such terms for the

whole sphere, and dividing by the volume of the sphere. Thus, since θ is a function of r,

$$\theta' = \frac{3}{4\pi R^3}\int_0^R \theta\, 4\pi r^2 dr \tag{45}$$

$$= \frac{6\theta_0}{\pi R^2}\left\{ e^{\frac{-\pi^2 h^2 t}{R^2}} \cdot \int_0^R r \sin\frac{\pi r}{R} dr - \frac{1}{2} e^{\frac{-4\pi^2 h^2 t}{R^2}} \cdot \int_0^R r \sin\frac{2\pi r}{R} dr + \frac{1}{3} e^{\frac{-9\pi^2 h^2 t}{R^2}} \cdot \int_0^R r \sin\frac{3\pi r}{R} dr - \cdots \right\} \tag{46}$$

$$= \frac{6\theta_0}{\pi^2}\left\{ e^{\frac{-\pi^2 h^2 t}{R^2}} + \frac{1}{4} e^{\frac{-4\pi^2 h^2 t}{R^2}} + \frac{1}{9} e^{\frac{-9\pi^2 h^2 t}{R^2}} + \cdots \right). \tag{47}$$

APPLICATIONS

133. Mercury Thermometer. Equation (47) may be applied to a spherical-bulb thermometer immersed in a stirred liquid. Neglecting the effect of the glass shell, the temperature of the mercury is given to a close approximation by the first term of (47) unless t is very small. The rate of cooling is

$$-\frac{\partial\theta'}{\partial t} = \frac{\pi^2 h^2}{R^2}\theta'. \tag{48}$$

134. Spherical Safes. Compare the fire-protecting qualities of two safes of solid steel ($h^2 = .121$) and solid concrete ($h^2 = .0058$), each spherical in form, of diameter 150 cm. (59″), and of very small internal cavity. Assuming that the surfaces are quickly raised from initial temperatures of 20° C. (68° F.) to 500° C. (932° F.), determine the temperatures at the centers after various periods of time.

Applying (44), we note that while it apparently takes an indeterminate form for $r = 0$, this is readily evaluated, since

$$\operatorname*{limit}_{r \doteq 0} \frac{\sin\frac{m\pi r}{R}}{\frac{m\pi r}{R}} = 1.$$

This gives

$$\theta = \theta_0\left(1 - 2\left\{e^{\frac{-\pi^2h^2t}{R^2}} - e^{\frac{-4\pi^2h^2t}{R^2}} + e^{\frac{-9\pi^2h^2t}{R^2}} - \cdots\right\}\right), \qquad (49)$$

from which we may compute that the temperature in the center of the steel safe would be 98° C. (208° F.) at the end of 1 hr., and 455° C. (850° F.) after 4 hr.; while in concrete the temperatures would run only 25° C. (77° F.) at the end of 10 hr., and not exceed 140° C. (284° F.) before 24 hr.

135. Steel Shot. Such a shot 3 cm. (1.18″) in diameter, at 800° C. (1472° F.), has its surface suddenly chilled to 20° C. (68° F.); what is the temperature 1 cm. below the surface in 1.8 sec.? Putting $r = .5$ and $R = 1.5$, also $h^2 = .121$ in (43), and making a suitable shift (20°) of the temperature scale, we readily find θ to be 501° C. (934° F.). It will be noted that the cooling is much more rapid than in the case treated in Art. 85.

The rate of cooling may be found by differentiating (43) with respect to t. This gives

$$\frac{\partial\theta}{\partial t} = -2\frac{\pi\theta_0 h^2}{Rr}\left\{\sin\frac{\pi r}{R}\cdot e^{\frac{-\pi^2h^2t}{R^2}} - 2\sin\frac{2\pi r}{R}\cdot e^{\frac{-4\pi^2h^2t}{R^2}} + \cdots\right\}. \qquad (50)$$

This equation might be used in an investigation of the relation between rapidity of cooling and hardness for approximately spherical steel ingots.

136. The preceding equations might be applied to a large number of practical problems of somewhat the same nature as those discussed in the last chapter. It is frequently desirable to know to what extent the temperature in the interior of mass of metal or other material departs from that at the surface, and by treating all roughly spherical shapes (for example, cubes) as spheres of the same volume, these equations may be applied at once. The theory might prove of service in such problems as the annealing of large steel castings, or in a study of the temperature stresses and consequent tendency to cracking which accompanies the quenching of large steel ingots.

PROBLEMS

1. The surface of a sphere of cinder concrete ($h^2 = .0031$) 30 cm. in diameter is rapidly raised to 1500° C. and held there. If it is all initially at zero, what will be the temperature of the center in 1 hr.? in 5 hr.? (48°; 1240°.)

2. A mercury thermometer of spherical bulb 1 cm. in diameter, at temperature 40° C., is immersed in a stirred mixture of ice and water. Neglecting the thickness of the glass, determine how soon, approximately, its average temperature is within .01° C. of that of the bath. (6.04 sec.)

3. Show from (44) or (47) that the common rule for roasting meats — of allowing so much time per pound, but decreasing somewhat this allowance per pound for the larger roasts — rests on a good theoretical basis.

Case IV

The Cooling of a Sphere by Radiation

137. We shall now solve a more difficult problem than any we have before attempted, namely, that of the temperature state in a sphere cooled by radiation. The solution will apply to the case of the sphere either in air or *in vacuo*, for the only assumption made in regard to the loss of heat is that Newton's law of cooling holds; that is, that the rate of loss of heat by a surface is proportional to the difference between its temperature and that of the surroundings.

As we shall see, the solution can also be applied to the case of a sphere of metal or other material of high conductivity, covered with a thin coating of some poorly conducting substance and placed in a bath at constant temperature. For the rate of loss of heat by the surface of the metal sphere will be proportional to the temperature gradient through the surface coating, that is, to the difference of temperature between the inner and outer surfaces of this coating, which, by the conditions of the problem, is equal to the difference of temperature of the metal surface and the bath. An example of this latter case is the mercury thermometer with spherical bulb, immersed in a liquid, it being desired to make correction for the glass envelope.

138. The differential equation for this case is, as before,

$$\frac{\partial (r\theta)}{\partial t} = h^2 \frac{\partial^2 (r\theta)}{\partial r^2}, \tag{51}$$

with the boundary conditions

$$\theta = f(r) \text{ when } t = 0, \tag{52}$$

$$-k\frac{\partial \theta}{\partial r} = E\theta \text{ at } r = R. \tag{53}$$

The last condition states that the rate at which heat is brought to unit area of the surface by conduction, namely, $-k\frac{\partial \theta}{\partial r}$, must be the rate at which it is radiated from this area, and this is $E\theta$, where E is the emissivity or coefficient of radiation of the surface. The surroundings are supposed to be at zero.

As before, put
$$u = r\theta; \tag{54}$$

then we have
$$\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial r^2}, \tag{55}$$

and the conditions
$$u = rf(r) \text{ when } t = 0, \tag{56}$$

$$u = 0 \text{ at } r = 0, \tag{57}$$

$$\frac{\partial u}{\partial r} + \left(C - \frac{1}{R}\right)u = 0 \text{ when } r = R, \tag{58}$$

where, for short, C is written for $\frac{E}{k}$.

Now we have already seen in Art. 66 that

$$u = e^{-h^2m^2t} \cos mr \tag{59}$$

and
$$u = e^{-h^2m^2t} \sin mr \tag{60}$$

are particular solutions of (55). Solution (59) is excluded by condition (57), but (60) satisfies this condition for all values of m. To see if (58) is also fulfilled, we substitute the value of u from (60) and get

$$mR \cos mR = (1 - CR) \sin mR. \tag{61}$$

If m_p is a root of this transcendental equation, then

$$u = e^{-h^2 m_p^2 t} \sin m_p r \tag{62}$$

is a particular solution of (55) satisfying (57) and (58). We must now endeavor to build up, with the aid of terms of the type (62), a solution which will also satisfy (56).

Since the sum of a number of particular solutions of a linear, homogeneous partial differential equation is also a solution, we note that

$$\begin{aligned} u = A_1 e^{-h^2 m_1^2 t} \cdot \sin m_1 r + A_2 e^{-h^2 m_2^2 t} \cdot \sin m_2 r \\ + A_3 e^{-h^2 m_3^2 t} \cdot \sin m_3 r + \cdots, \end{aligned} \tag{63}$$

where $A_1, A_2, A_3, \cdots$, are arbitrary constants, is a solution of (55) satisfying (57). It moreover satisfies (58) if $m_1, m_2, m_3, \cdots$, are roots of equation (61). It evidently reduces for $t = 0$ to

$$A_1 \sin m_1 r + A_2 \sin m_2 r + A_3 \sin m_3 r + \cdots, \tag{64}$$

and if it is possible to develop $rf(r)$, for all values of r between 0 and R, in terms of such a series, we shall have (56) satisfied as well.

139. The solution of our problem, then, will consist of two parts: first, the solution of the transcendental equation (61), that is, the determination of the roots $m_1, m_2, m_3, \cdots$ (we anticipate a fact shortly to be shown, namely, that there are an infinite number of such roots); and second, the expansion of the function $rf(r)$ in the sine series (64). The second part of the problem is analogous to development in terms of a Fourier's series, but more complicated because the numbers m_1, m_2, m_3, instead of being the integers 1, 2, 3, as in the regular Fourier's series, must in the present case be roots of equation (61).*

140. The Solution of the Transcendental Equation. The roots of equation (61) are easily obtained by computation, but a study of their values under various conditions may be most easily made by graphical methods. If we make the substitutions

$$\alpha = mR \tag{65}$$

*This is the most general sine development which can be obtained by Fourier's method. See Byerly, *Fourier's Series and Spherical Harmonics*, p. 121.

and
$$\beta = 1 - CR, \tag{66}$$
equation (61) becomes
$$\alpha \cos \alpha = \beta \sin \alpha, \tag{67}$$
or, more simply,
$$\alpha = \beta \tan \alpha. \tag{68}$$

Then if we construct the curves
$$y = \tan x \tag{69}$$
and
$$y = \frac{x}{\beta}, \tag{70}$$
their points of intersection will give the values of x for which
$$\frac{x}{\beta} = \tan x; \tag{71}$$
that is, the roots of (68) and hence of (61).

141. We may draw some general conclusions as to these roots. In the first place there are evidently an infinite number of positive roots, and the same number of negative, which are equal in absolute value to the positive. The values of the roots vary between certain limits with the slope of the line $y = \frac{x}{\beta}$, that is, with the value of C, or $\frac{E}{k}$. Since C can have, theoretically at least, any value between 0 and ∞ but must always be positive, the slope
$$\frac{1}{\beta} = \frac{1}{1 - CR} \tag{72}$$
can have any value between 1 and ∞ or between 0 and $-\infty$.

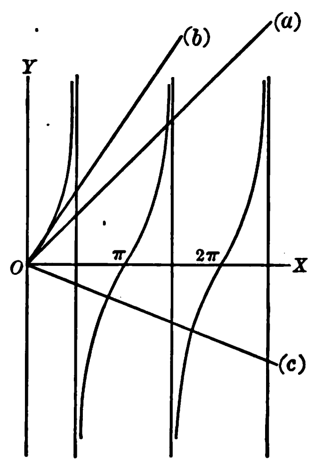


FIG. 24

We can easily show with the aid of a figure the approximate values of the roots for the several cases as follows:

Let $C = 0$, corresponding to the case of a sphere protected with a thermally impervious covering. The roots then correspond to the intersections of the line (a) (Fig. 24) of 45° slope.

Their values are $0, \alpha_1, \alpha_2, \cdots$, where

$$\pi < \alpha_1 < \frac{3\pi}{2}; \; 2\pi < \alpha_2 < \frac{5\pi}{2}; \; \cdots \; n\pi < \alpha_n < \left(n + \frac{1}{2}\right)\pi; \quad (73)$$

α_n in this case approaches the limit $(n + \frac{1}{2})\pi$ as n increases.

Next let C lie between 0 and $\frac{1}{R}$ so that $0 < (1 - CR) < 1$. The line (b) corresponds to this case, and the roots $0, \alpha_1, \alpha_2, \alpha_3, \cdots$, have the values

$$0 < \alpha_1 < \frac{\pi}{2}; \; \pi < \alpha_2 < \frac{3\pi}{2}; \; \cdots (n-1)\pi < \alpha_n < \left(n - \frac{1}{2}\right)\pi, \quad (74)$$

approaching the larger values as C increases. When $C = \frac{1}{R}$, then the roots become

$$0, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots. \quad (75)$$

Finally, if C lies between $\frac{1}{R}$ and ∞, the intersecting straight line will fall below the axis in some position such as (c), and the roots $0, \alpha_1, \alpha_2, \cdots$, will have values

$$\frac{\pi}{2} < \alpha_1 < \pi; \; \frac{3\pi}{2} < \alpha_2 < 2\pi; \cdots \left(n - \frac{1}{2}\right)\pi < \alpha_n < n\pi \cdots \quad (76)$$

which become for $C = \infty$

$$\alpha_1 = \pi, \; \alpha_2 = 2\pi, \; \cdots, \; \alpha_n = n\pi \cdots. \quad (77)$$

From these roots $\alpha_1, \alpha_2, \alpha_3$, the values $m_1, m_2, m_3, \cdots$, satisfying equation (61) are obtained at once with the aid of (65).

142. The General Sine Series Development. We shall arrive at this development by assuming that it is possible to expand $rf(r)$ in a series

$$\begin{aligned} rf(r) = A_1 \sin m_1 r + A_2 \sin m_2 r + \cdots \\ + A_b \sin m_b r + \cdots \equiv \sum_{b=1}^{b=\infty} A_b \sin m_b r, \end{aligned} \quad (78)$$

just as we assumed before that such a function could be expanded in an ordinary Fourier's series, and then proceed to find the values of the coefficients $A_1, A_2, A_3, \cdots$, to which this assumption leads. The values $m_1, m_2, m_3, \cdots$, are the roots of equation

(61) determined above. While zero is a root in each case, there is no corresponding term in the series since $\sin 0 = 0$. The negative roots which occur are included with the positive in the terms of (78), for since $\sin(-x) = -\sin x$, we may write

$$A_b' \sin m_b r + A_b'' \sin(-m_b r) = A_b \sin m_b r. \tag{79}$$

Multiplying each side of equation (78) by $\sin m_a r dr$ and integrating from 0 to R,

$$\int_0^R r f(r) \sin m_a r dr = \sum_{b=1}^{b=\infty} A_b \int_0^R \sin m_b r \cdot \sin m_a r dr. \tag{80}$$

Now $\displaystyle\int_0^R \sin m_b r \cdot \sin m_a r dr$

$$= \frac{1}{2}\int_0^R \{\cos[(m_b - m_a)r] - \cos[(m_b + m_a)r]\} dr \tag{81}$$

$$= \frac{\sin[(m_b - m_a)R]}{2(m_b - m_a)} - \frac{\sin[(m_b + m_a)R]}{2(m_b + m_a)} \tag{82}$$

$$= \frac{\{m_a \sin m_b R \cdot \cos m_a R - m_b \cos m_b R \cdot \sin m_a R\}}{(m_b^2 - m_a^2)}. \tag{83}$$

But since m_a and m_b are roots of (61),

$$m_a R = (1 - CR)\tan m_a R; \quad m_b R = (1 - CR)\tan m_b R, \tag{84}$$

so that

$$m_a \tan m_b R = m_b \tan m_a R, \tag{85}$$

or

$$m_a \sin m_b R \cdot \cos m_a R = m_b \sin m_a R \cdot \cos m_b R. \tag{86}$$

Therefore

$$\int_0^R \sin m_b r \cdot \sin m_a r dr = 0, \tag{87}$$

when m_a and m_b are different. If they are equal, we have

$$\int_0^R \sin^2 m_a r dr = \frac{1}{2}\int_0^R [1 - \cos(2 m_a r)] dr \tag{88}$$

$$= \frac{R}{2} - \frac{\sin 2 m_a R}{4 m_a}. \tag{89}$$

Now

$$\sin 2 m_a R = \frac{2 \tan m_a R}{1 + \tan^2 m_a R} \tag{90}$$

$$= \frac{2 m_a R(1 - CR)}{(CR - 1)^2 + m_a^2 R^2}. \tag{91}$$

Therefore $$\int_0^R \sin^2 m_a r dr = \frac{R}{2}\frac{m_a^2R^2 + CR(CR-1)}{m_a^2R^2+(CR-1)^2}. \tag{92}$$

Applying this in the series (80), that is, in

$$\int_0^R rf(r)\sin m_a r dr = A_1\int_0^R \sin m_1 r\cdot\sin m_a r dr$$
$$+ A_2\int_0^R \sin m_2 r\cdot \sin m_a r dr + \cdots, \tag{93}$$

we have

$$A_a = \frac{2}{R}\frac{m_a^2R^2+(CR-1)^2}{m_a^2R^2+CR(CR-1)}\cdot\int_0^R rf(r)\sin m_a r dr. \tag{94}$$

143. Final Solution. Our problem is now solved, for we have evaluated the coefficients of the series (78) in terms of the roots of equation (61), which roots we have shown to have real values which are easily determined. The solution may be written

$$u = \sum_{a=1}^{a=\infty} A_a e^{-h^2m_a^2t}\cdot \sin m_a r, \tag{95}$$

or, evaluating A_a from (94), and remembering that $u = r\theta$,

$$\theta = \frac{2}{rR}\sum_{a=1}^{a=\infty}\frac{m_a^2R^2+(CR-1)^2}{m_a^2R^2+CR(CR-1)}\cdot e^{-h^2m_a^2t}\cdot\sin m_a r$$
$$\int_0^R \lambda f(\lambda)\sin m_a\lambda d\lambda. \tag{96}$$

144. Initial Temperature θ_0. In the case in which the initial temperature of the sphere is everywhere the same, that is, $f(r) = \theta_0$, we find that

$$\theta_0\int_0^R \lambda\sin m\lambda d\lambda = \frac{\theta_0}{m^2}\{\sin mR - mR\cos mR\} \tag{97}$$

and, with the use of (61), $$= \frac{CR\theta_0}{m^2}\sin mR. \tag{98}$$

So that (96) becomes for this case

$$\theta = \frac{2C\theta_0}{r}\left\{\frac{m_1^2R^2+(CR-1)^2}{m_1^2[m_1^2R^2+CR(CR-1)]}\cdot e^{-h^2m_1^2t}\cdot\sin m_1R\cdot\sin m_1 r\right.$$
$$\left.+\frac{m_2^2R^2+(CR-1)^2}{m_2^2[m_2^2R^2+CR(CR-1)]}\cdot e^{-h^2m_2^2t}\cdot\sin m_2R\cdot\sin m_2r+\cdots\right\}. \tag{99}$$

145. Special Cases. If CR is small in comparison with unity, as it would be in many cases, the problem is greatly simplified. For an inspection of Fig. 24 shows that in this case m_1R will be very small, while the other values of mR will be larger than π, so that only the first term of the series (99) need be considered. The value of m_1 is readily determined from (61) by developing the sine and cosine in series and neglecting higher powers of m_1R, in which case we obtain

$$1 - \tfrac{1}{2} m_1^2R^2 = (1 - CR)(1 - \tfrac{1}{6} m_1^2R^2), \tag{100}$$

from which it follows that

$$m_1^2 = \frac{3\,C}{R}. \tag{101}$$

With the aid of (101), equation (99) may be still further simplified if it be remembered that m_1R and m_1r are small quantities, and if C^2R^2 is neglected, for it reduces at once to

$$\theta = \theta_0 e^{\frac{-3\,Ch^2t}{R}} \tag{102}$$

$$= \theta_0 e^{\frac{-3\,Et}{\rho cR}}, \tag{103}$$

c being the specific heat.

146. The assumptions involved in this last formula are that the sphere is so small or the cooling so slow that the temperature at any time is sensibly uniform throughout the whole volume. With this assumption it may be derived independently in a very simple manner; for the quantity of heat which the sphere radiates in time dt is

$$4\,\pi R^2E\theta dt. \tag{104}$$

This means a change in temperature of the sphere of $d\theta$, which corresponds to a quantity of heat given up equal to

$$-\tfrac{4}{3}\,\pi R^3c\rho d\theta, \tag{105}$$

the negative sign being used, since $d\theta$ is a negative quantity. Hence we have

$$4\,\pi R^2E\theta dt = -\tfrac{4}{3}\,\pi R^3c\rho d\theta, \tag{106}$$

the integration of which gives, since the temperature of the sphere is θ_0 at the time $t = 0$,

$$\theta = \theta_0 e^{\frac{-3Et}{c\rho R}}, \tag{107}$$

as above.

147. Applications. Equations (96) or (99) make possible the treatment of the problem of the cooling of the earth by radiation before the formation of a surface crust, which was kept, by the evaporation of the water, at a nearly constant temperature. The solutions of Cases III and IV of the present chapter would enable one to treat the problem of terrestrial temperatures with account taken of the spherical shape of the earth, but as already noted our present data would by no means warrant such a rigorous solution, which would alter the result in any case by only a very small fraction. It may be noted that the solution of the problem of radiation for the semi-infinite solid is gained from the present case by letting R approach infinity.

As already suggested, the solution for the present case will fit another which at first sight seems quite foreign to it, namely, the cooling of a mercury-in-glass thermometer in a liquid. If the glass is so thin, as it usually is, that its heat capacity can be neglected, we have only to set in place of E, in the above equations, $\frac{k}{l}$, where l is the thickness of the glass and k its conductivity, and we shall have a solution of this problem.

148. An experimental method of determining the conductivity of poor conductors has been based on the solution of the problem of the radiating sphere, and used by Ayrton and Perry * in determining the conductivity of stone. The series (95) converges so rapidly that after a sufficient length of time the terms after the first may be neglected. The temperature is then given by

$$\theta = A_1 e^{-k^2 m_1^2 t} \cdot \frac{\sin m_1 r}{r}, \tag{108}$$

and for the center of the sphere this becomes

$$\theta_c = A_1 m_1 e^{-k^2 m_1^2 t}. \tag{109}$$

* *Phil. Mag.* (5), **5**, p. 241 (1878); see Carslaw's *Fourier's Series and Integrals*, p. 336.

The coefficient A may be determined just as before by multiplying each side of equation (109) by $\sin m_1 r dr$ and integrating from 0 to R, when $t = 0$. Then

$$A_1 = \theta_0 \frac{\int_0^R r \sin m_1 r dr}{\int_0^R \sin^2 m_1 r dr} \tag{110}$$

$$= \frac{2\,\theta_0}{m_1}\left(\frac{\sin m_1 R - m_1 R \cos m_1 R}{m_1 R - \sin m_1 R \cdot \cos m_1 R}\right). \tag{111}$$

Therefore $$\theta_c = 2\,\theta_0 \frac{\sin m_1 R - m_1 R \cos m_1 R}{m_1 R - \sin m_1 R \cdot \cos m_1 R} e^{-h^2 m_1^2 t} \tag{112}$$

$$= Ne^{-nt}, \text{ say.} \tag{113}$$

The value of n can be obtained from observations of the temperature at the center at two different times. N can then be found, knowing n, and from a table of values of the expression

$$\frac{\sin x - x \cos x}{x - \sin x \cos x}, \tag{114}$$

the value of m_1 can be determined from the known value of N. Then since $n = h^2 m_1^2$, the diffusivity h is determined, and from this, knowing the specific heat and density, the conductivity.

PROBLEMS

1. A wrought-iron cannon ball of 10 cm. radius and at a uniform temperature of 50° C. is allowed to cool by radiation in a vacuum to surroundings at 30° C. If the value of E for the surface is .00015 cal. per sec. per square centimeter per degree, what will be the temperature at the center and at the surface after 1 hr.? (46.3°; 46.2°.)

2. A thermometer with spherical mercury bulb of 3.5 mm. outside, and 2.5 mm. inside radius, heated to an initial temperature of 30° C., is plunged into stirred ice water. Find, to a first approximation, how long before the temperature at its center will fall to within $\frac{1}{4}$° of that of the bath. Neglect the heat capacity, but not the conductivity of the glass. (7.5 sec.)

3. The initial temperature of an orange 10 cm. in diameter is 15° C. while the surroundings are at 0° C. If the emissivity of the surface is .00025 and the thermal constants of the orange the same as those of water, what will be the temperature 1 cm. below the surface after 8 hr.? (.38°.)

Case V

General Case of Heat Flow in an Infinite Medium

149. In Case II of this chapter we solved the problem of the flow of heat from an instantaneous point source. We shall extend this result to cover the case in which we have an initial arbitrary distribution of heat, the initial temperature being given as a function of the coördinates in three dimensions.

Let x, y, z, be the coördinates of any point whose temperature we wish to investigate at any time t, while λ, μ, ν, are the coördinates of any heated element of volume, and become in general the variables of integration. Then the initial temperature is

$$\theta_0 = f(\lambda, \mu, \nu), \tag{115}$$

and the quantity of heat initially contained in any volume element $d\lambda d\mu d\nu$ is

$$dq = f(\lambda, \mu, \nu)\frac{k}{h^2}d\lambda d\mu d\nu. \tag{116}$$

If this quantity of heat is propagated through the body, it will produce a rise in temperature which can be obtained at once from (23), and which is, since

$$r^2 = (\lambda - x)^2 + (\mu - y)^2 + (\nu - z)^2, \tag{117}$$

$$d\theta = \left(\frac{1}{2\,h\sqrt{\pi t}}\right)^3 e^{\frac{-[(\lambda - x)^2 + (\mu - y)^2 + (\nu - z)^2]}{4h^2 t}} \cdot f(\lambda, \mu, \nu)\,d\lambda d\mu d\nu. \tag{118}$$

The temperature at any point will be the sum of all these increments of temperature and may be obtained by integrating (118)

$$\theta = \left(\frac{1}{2\,h\sqrt{\pi t}}\right)^3 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\frac{-[(\lambda - x)^2 + (\mu - y)^2 + (\nu - z)^2]}{4h^2 t}} \cdot f(\lambda, \mu, \nu)\,d\lambda d\mu d\nu. \tag{119}$$

Making the substitutions

$$\alpha = \frac{\lambda - x}{2\,h\sqrt{t}};\ \beta = \frac{\mu - y}{2\,h\sqrt{t}};\ \gamma = \frac{\nu - z}{2\,h\sqrt{t}}, \tag{120}$$

this becomes

$$\theta = \frac{1}{\pi^{\frac{3}{2}}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\alpha^2-\beta^2-\gamma^2} f(x + 2\,h\alpha\sqrt{t},\ y + 2\,h\beta\sqrt{t},\ z + 2\,h\gamma\sqrt{t})\,d\alpha d\beta d\gamma. \tag{121}$$

150. It will be instructive to show how this solution may be obtained independently as a particular integral of the conduction equation

$$\frac{\partial \theta}{\partial t}=h^2\left\{\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right\}, \tag{122}$$

subject to the boundary condition

$$\theta=f(\lambda,\mu,\nu) \text{ when } t=0. \tag{123}$$

Assume $\theta=XYZ$, where X is a function of x and t, and where Y and Z are functions of y, t and z, t respectively. Then we have from (122)

$$YZ\frac{\partial X}{\partial t}+XZ\frac{\partial Y}{\partial t}+XY\frac{\partial Z}{\partial t}$$
$$=h^2\left\{YZ\frac{\partial^2 X}{\partial x^2}+XZ\frac{\partial^2 Y}{\partial y^2}+XY\frac{\partial^2 Z}{\partial z^2}\right\}. \tag{124}$$

But since X, Y, and Z are essentially independent, being functions of the independent variables x, y, z, this can only be true if the corresponding terms on each side of the equation are equal, that is, if

$$\frac{\partial X}{\partial t}=h^2\frac{\partial^2 X}{\partial x^2}, \tag{125}$$

with similar equations for Y and Z.

Now it may be easily shown by differentiation that

$$X=\frac{1}{\sqrt{t}}e^{\frac{-(\lambda-x)^2}{4h^2t}} \tag{126}$$

is a particular solution of (125), — a type of solution already made use of in Art. 93, — so that

$$\theta=\frac{1}{\sqrt{t}}e^{\frac{-(\lambda-x)^2}{4h^2t}}\cdot\frac{1}{\sqrt{t}}e^{\frac{-(\mu-y)^2}{4h^2t}}\cdot\frac{1}{\sqrt{t}}e^{\frac{-(\nu-z)^2}{4h^2t}} \tag{127}$$

is an integral of (122). Therefore if C is any constant, and $\phi(\lambda,\mu,\nu)$ an arbitrary function of λ, μ, ν,

$$\theta=C\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\frac{1}{\sqrt{t}}\right)^3\cdot e^{\frac{-[(\lambda-x)^2+(\mu-y)^2+(\nu-z)^2]}{4h^2t}}\cdot\phi(\lambda,\mu,\nu)\,d\lambda d\mu d\nu \tag{128}$$

is also a solution of (122). By the substitutions (120) this reduces to

$$\theta = C(2h)^3 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\alpha^2-\beta^2-\gamma^2}\phi(x+2h\alpha\sqrt{t},$$
$$y+2h\beta\sqrt{t},\ z+2h\gamma\sqrt{t})d\alpha d\beta d\gamma. \quad (129)$$

If we now let $t = 0$, this becomes

$$\theta_0 = C(2h)^3 \cdot \phi(x, y, z)\int_{-\infty}^{+\infty} e^{-\alpha^2}d\alpha \int_{-\infty}^{+\infty} e^{-\beta^2}d\beta \int_{-\infty}^{+\infty} e^{-\gamma^2}d\gamma, \quad (130)$$

and, remembering that

$$\int_{-\infty}^{+\infty} e^{-q^2}dq = \sqrt{\pi}, \quad (131)$$

this becomes

$$\theta_0 = C(2h\sqrt{\pi})^3\phi(x, y, z). \quad (132)$$

From (123) we see that if

$$C = \left(\frac{1}{2h\sqrt{\pi}}\right)^3 \quad (133)$$

and $\phi(x, y, z) = f(x, y, z) = f(\lambda, \mu, \nu)$, since $t = 0$ (134)

the boundary condition (123) is fulfilled. Putting in (129) these values of C and ϕ, we find at once that it reduces to the solution (121) already found.

PROBLEMS

1. Molten copper at 1085° C. is suddenly poured into a cubical cavity in a large mass of copper at 0° C. If the edge of the cube is 40 cm., find the temperature at the center after 15 min. Neglect the latent heat of fusion. (Compare Problem 1, p. 129.) (43°.)

2. Solve the problem of the *steady state* of temperature in a long rod, one half of whose surface (that is, one half the circumference of each section) is kept at θ_0 and the other at zero.

The Fourier equation for cylindrical coördinates becomes in this case, for the steady state,

$$r\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right) + \frac{\partial^2\theta}{\partial\phi^2} = 0.$$

If the radius of the rod is R, the surface temperature being θ_0 for $0 < \phi < \pi$, and zero for $\pi < \phi < 2\pi$, then the temperature at any point will be found to be given by

$$\theta = \frac{\theta_0}{2} + \frac{2\theta_0}{\pi}\left\{\frac{r}{R}\sin\phi + \frac{r^3}{R^3}\frac{\sin 3\phi}{3} + \frac{r^5}{R^5}\frac{\sin 5\phi}{5} + \cdots\right\}.$$

CHAPTER IX

THE FORMATION OF ICE

151. We shall now take up the study of the formation of ice, that is, of the relationship which must exist between the thickness and rate of freezing or melting of a sheet of ice and the time when a lake of still water is frozen or a sheet of ice thawed. In our previous study of the various cases of heat conduction in a medium we have assumed that the addition or subtraction of heat from any element of the medium serves only to change its temperature and does not in any way alter its conductivity constants or other physical properties. In ice formation, however, we have essentially a more complicated case, for the freezing of water or thawing of ice results not only in a change from one medium to another which has entirely different thermal constants, but also in the accompanying release or absorption of the latent heat of fusion.

152. We shall treat the problem in two somewhat different ways, the first following substantially the method of Franz Neumann * and the second that of J. Stefan.† In each case we have initially a surface of still water lowered, as by contact with the air or some other body, to some temperature θ_0, which must always be below the freezing point. There will then be formed a layer of ice whose thickness ϵ is a function of the time t. Take the upper surface of ice as the yz-plane, and the positive x-direction as running into the ice. Let θ_1 apply to temperatures in the ice, and θ_2 to the water; and similarly, let k_1, c_1, h_1^2, be the thermal constants for ice, while k_2, c_2, h_2^2, are those for water. It is assumed that there is no convection in the water, and the changes of volume which occur on freezing or melting are neglected.

* Weber-Riemann, *Part. Diff. Gleichungen*, II, p. 117.

† *Wied. Ann.*, **42**, p. 269. See also Tamura, *Monthly Weather Review*, February, 1905.

153. Neumann's Solution. Instead of one fundamental equation, as in the case of a single homogeneous medium, there will now be two, applying respectively to the ice and to the water under the ice. These are

$$\frac{\partial \theta_1}{\partial t} = h_1^2 \frac{\partial^2 \theta_1}{\partial x^2} \text{ in the ice, } (0 < x < \epsilon), \tag{1}$$

and

$$\frac{\partial \theta_2}{\partial t} = h_2^2 \frac{\partial^2 \theta_2}{\partial x^2} \text{ in the water } (\epsilon < x). \tag{2}$$

The temperature of the boundary surface of ice and water (at $x = \epsilon$) must always be $0°$ C. and there will be continual formation of new ice. If the thickness increases by $d\epsilon$ in time dt, there will be set free for each unit of area an amount of heat

$$Q = L\rho d\epsilon, \tag{3}$$

where L is the latent heat of fusion. This must escape upward by conduction through the ice, and in addition there will be a certain amount of heat carried away from the water below, so that the total amount of heat which flows outward through unit area of the lower surface of the ice sheet is

$$Q' = k_1 \left(\frac{\partial \theta_1}{\partial x}\right)_{x=\epsilon} dt. \tag{4}$$

Of this amount the quantity

$$Q'' = k_2 \left(\frac{\partial \theta_2}{\partial x}\right)_{x=\epsilon} dt \tag{5}$$

flows up from the water below; hence we obtain for our first boundary condition

$$\left(k_1 \frac{\partial \theta_1}{\partial x} - k_2 \frac{\partial \theta_2}{\partial x}\right)_{x=\epsilon} = L\rho \frac{\partial \epsilon}{\partial t}. \tag{6}$$

The other boundary conditions are to be

$$\theta_1 = \theta_0 = C_1 \text{ at } x = 0 \tag{7}$$

$$\theta_1 = \theta_2 = 0 \text{ at } x = \epsilon \tag{8}$$

$$\theta_2 = C_2 \text{ at } x = \infty. \tag{9}$$

We also have three other boundary conditions derived from the fact that when $t = 0$, ϵ is fixed, while θ_1 and θ_2 must be given

as functions of x, the first between the limits 0 and ϵ and the last between ϵ and ∞. We shall investigate later the particular form of these functions.

154. A general solution of the problem for these conditions is not possible as yet, for the condition (6) containing the unknown function ϵ is not linear and homogeneous, and we cannot then expect to reach a solution by the combination of particular solutions. Our method of solution then will be to seek particular integrals of (1) and (2), and after modifying them to fit boundary conditions (7), (8), and (9), find under what conditions the solution will satisfy (6). This will then determine the initial values of e, θ_1, and θ_2.

From equation (35) of Art. 76 we can conclude that if we define the function $\Theta(x)$ by the equation

$$\Theta(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-\beta^2} d\beta\, *, \tag{10}$$

then $\Theta\left(\frac{x}{2h\sqrt{t}}\right)$ is a solution of the differential equations (1) and (2). Consequently, if A_1, B_1, A_2, B_2, are constants,

$$\theta_1 = A_1 + B_1 \Theta\left(\frac{x}{2h_1\sqrt{t}}\right) \tag{11}$$

and

$$\theta_2 = A_2 + B_2 \Theta\left(\frac{x}{2h_2\sqrt{t}}\right) \tag{12}$$

are also solutions. Now boundary condition (8) means that $\Theta\left(\frac{\epsilon}{2h\sqrt{t}}\right)$ must be constant, which will be true if $\epsilon = 0$, $\epsilon = \infty$, or if ϵ is proportional to $\sqrt{t}$. The first two of these assumptions are evidently inconsistent with (8), so there remains only the last, which we may put in the form

$$\epsilon = \alpha\sqrt{t}, \tag{13}$$

where α is a constant we shall determine later, together with A_1, B_1, A_2, and B_2.

* See Appendix D.

From the properties of the integral (10) we know that $\Theta(0)=0$ and $\Theta(\infty)=1$. Then fitting boundary conditions (7), (8), and (9) in (11) and (12) with the use of (13), we find that

$$A_1 = C_1, \tag{14}$$

$$A_1 + B_1\Theta\left(\frac{\alpha}{2\,h_1}\right) = 0, \tag{15}$$

$$A_2 + B_2\Theta\left(\frac{\alpha}{2\,h_2}\right) = 0, \tag{16}$$

$$A_2 + B_2 = C_2, \tag{17}$$

while (11), (12), and (13) in connection with (6) give

$$\frac{k_1B_1}{h_1\sqrt{\pi t}}e^{\frac{-\alpha^2}{4h_1^2}} - \frac{k_2B_2}{h_2\sqrt{\pi t}}e^{\frac{-\alpha^2}{4h_2^2}} = \frac{L\rho\alpha}{2\sqrt{t}}. \tag{18}$$

Solving equations (14)–(17) for B_1 and B_2, we get

$$B_1 = \frac{-C_1}{\Theta\left(\frac{\alpha}{2\,h_1}\right)};\quad B_2 = \frac{C_2}{1-\Theta\left(\frac{\alpha}{2\,h_2}\right)}; \tag{19}$$

and substituting these values in (18), we have finally

$$\frac{k_1C_1e^{\frac{-\alpha^2}{4h_1^2}}}{h_1\Theta\left(\frac{\alpha}{2\,h_1}\right)} + \frac{k_2C_2e^{\frac{-\alpha^2}{4h_2^2}}}{h_2\left[1-\Theta\left(\frac{\alpha}{2\,h_2}\right)\right]} = -L\rho\alpha\frac{\sqrt{\pi}}{2}. \tag{20}$$

155. This transcendental equation can be solved for α by the method employed in Art. 140. Plot the curves

$$y = -L\rho\frac{\sqrt{\pi}}{2}\cdot\alpha, \tag{21}$$

and

$$y = f(\alpha), \tag{22}$$

where $f(\alpha)$ represents the left-hand side of (20). Then α is given as the abscissa of the intersection of the two curves. Having found α, the problem is solved, for from (13) we can then express the exact relation between the thickness and time, and having solved (14)–(17) for A_1, B_1, A_2, and B_2, we have from (11) and (12) the temperatures at any point in the water or ice.

156. We are now able to specify the initial conditions for which we have solved the problem, and which have up to this time been indeterminate. It follows from (13) that when $t = 0$, $\epsilon = 0$, and from (12) that θ_2 is initially equal to $A_2 + B_2 = C_2$, everywhere except at the point $x = 0$, where it is indeterminate. This means that we have taken the instant $t = 0$ as that at which the ice just begins to form, the water being everywhere at the constant temperature C_2. Inasmuch, then, as there is no ice at time $t = 0$, the temperature θ_1 must be indeterminate, as is shown by (11).

157. In the case of freezing as just treated, C_1 is necessarily a negative and C_2 a positive quantity. By reversing the signs and making C_1 positive and C_2 negative we have equations applicable to thawing. But thawing in this case means that a layer of water is formed on the ice and that the heat flows in from the upper surface of the water, which is then at temperature C_1. But this means that the ice and water have just changed places, so that in the case of thawing, C_1, k_1, h_1^2, and c_1 apply to the water, while C_2, k_2, h_2^2, and c_2 apply to the ice.

158. Stefan's Solution. Stefan simplified the conditions of the problem by assuming that the temperature of the water was everywhere constant and equal to zero. The fundamental equation (1) then becomes

$$\frac{\partial \theta_1}{\partial t} = h_1^2 \frac{\partial^2 \theta_1}{\partial x^2} \text{ for } 0 < x < \epsilon, \tag{23}$$

while the second is missing. Likewise the boundary conditions (6)–(9) are simplified to

$$k_1 \left[\frac{\partial \theta_1}{\partial x}\right]_{x=\epsilon} = L\rho \frac{\partial \epsilon}{\partial t}, \tag{24}$$

$$\theta_1 = \theta_0 = C_1 \text{ at } x = 0, \tag{25}$$

$$\theta_1 = 0 \quad \text{at } x = \epsilon. \tag{26}$$

As θ_1 may be expressed as a function of both time and place, we may write its total differential

$$d\theta_1 = \frac{\partial \theta_1}{\partial x} dx + \frac{\partial \theta_1}{\partial t} dt. \tag{27}$$

From (26) we see that this total differential must be zero at $x=\epsilon$, so that

$$\left[\frac{\partial\theta_1}{\partial t}\right]_{x=\epsilon}+\left[\frac{\partial\theta_1}{\partial x}\right]_{x=\epsilon}\cdot\frac{\partial\epsilon}{\partial t}=0, \tag{28}$$

so that with the aid of (24) we have

$$\left[\frac{\partial\theta_1}{\partial t}\right]_{x=\epsilon}=-\frac{h_1^2c_1}{L}\left[\frac{\partial\theta_1}{\partial x}\right]^2_{x=\epsilon}, \text{ since } k=h^2cp. \tag{29}$$

As a special solution of (23) we shall examine the integral

$$\theta=A\int_{\frac{x}{2h\sqrt{t}}}^{\beta}e^{-z^2}dz \tag{30}$$

and see if the constants A and β can be so chosen that this solution is consistent with the conditions (24), (25), (26), and (28). We need not prove that (30) is a particular integral of (23), for in Chapter VII we have used this type of integral many times as a solution of the Fourier equation in one dimension, so we can proceed at once with our attempt at fitting it to these boundary conditions.

Condition (25) demands that

$$C_1=A\int_0^{\beta}e^{-z^2}dz, \tag{31}$$

which gives one relation between A and β. Condition (26) means that the two limits of the integral must be the same for $x=\epsilon$, so that

$$\beta=\frac{\epsilon}{2h\sqrt{t}} \text{ or } \epsilon=2\beta h\sqrt{t}. \tag{32}$$

This gives the same law of thickness as found by Neumann's method (13), namely, that the thickness increases with the square root of the time. However, we have not yet determined the constant β, and to do this we must use (29). The differential coefficients $\frac{\partial\theta_1}{\partial t}$ and $\frac{\partial\theta_1}{\partial x}$ are obtained from (30) after the method described in Art. 79, and are

$$\frac{\partial\theta_1}{\partial t}=Ae^{\frac{-x^2}{4h_1^2t}}\cdot\frac{x}{4th_1\sqrt{t}}, \tag{33}$$

$$\frac{\partial\theta_1}{\partial x} = -Ae^{\frac{-x^2}{4h_1^2t}} \cdot \frac{1}{2h_1\sqrt{t}}. \tag{34}$$

If we now put in these expressions $x = \epsilon = 2\beta h_1\sqrt{t}$ and then apply (29), we have

$$Ae^{-\beta^2} \cdot \frac{\beta}{2t} = -\frac{h_1^2c_1}{L}A^2e^{-2\beta^2} \cdot \frac{1}{4h_1^2t}; \tag{35}$$

or, with the use of (31),

$$\beta e^{\beta^2}\int_0^\beta e^{-z^2}dz = -\frac{C_1c_1}{2L}, \tag{36}$$

and this equation enables us to determine β. The integral may be evaluated by expanding e^{-z^2} in the customary power series and performing the integration. When this result is multiplied by the series for βe^{β^2}, we get a series whose first two terms are

$$\beta^2\left(1 + \frac{2\beta^2}{3}\right). \tag{37}$$

To a first approximation, then, (36) gives

$$\beta^2 = -\frac{C_1c_1}{2L}; \tag{38}$$

consequently, to the same degree of approximation, (32) means that

$$\epsilon^2 = -\frac{2C_1c_1h_1^2t}{L}. \tag{39}$$

For the second approximation

$$\beta^2\left(1 + \frac{2\beta^2}{3}\right) = -\frac{C_1c_1}{2L}, \tag{40}$$

from which β, and consequently ϵ, is readily determined.

Since C_1 is intrinsically negative, the right-hand member of the above equation is a positive quantity.

It should be noted that the same law of freezing holds in each case, that is, the proportionality of thickness with the square root of the time; the proportionality constant only is changed. Indeed if we put $C_2 = 0$ in Neumann's solution (20), it reduces at once to Stefan's solution (36), if $\alpha = 2\beta h$. This makes the two

expressions for the thickness, (13) and (32), identical, and shows that Stefan's solution may be regarded as only a special case of Neumann's.

159. Thickness of Ice Proportional to Time. Stefan also outlined the solution of one or two special cases which we shall find interesting.

Consider the expression

$$\theta_1 = \frac{A}{p}(e^{pt-qx} - 1), \tag{41}$$

where A, p, and q are constants.

It may be readily seen upon differentiation that if

$$p = h_1^2 q^2, \tag{42}$$

(41) is a solution of the fundamental equation (23). Now

$$\theta_1 = 0 \text{ for } pt - qx = 0; \tag{43}$$

and from (26) $$\theta_1 = 0 \text{ at } x = \epsilon, \tag{44}$$

from which $$pt - qx = 0 \text{ at } x = \epsilon \tag{45}$$

or $$\epsilon = qh_1^2 t. \tag{46}$$

This shows that the thickness of ice *may* increase in direct proportion to the time if θ_0 is not a constant, as we have heretofore taken it. Equation (41) shows that (since $\theta_1 = \theta_0$ when $x = 0$) θ_0 must be a function of the time, and it will be our task to investigate the form of this function.

Since (29) must hold, we find on substitution of (41) and (46) that

$$A = -\frac{c_1 h_1^2 q^2 A^2}{p^2 L} = -\frac{c_1 A^2}{pL}, \tag{47}$$

so that the relation between A and p is

$$p = -\frac{Ac_1}{L}. \tag{48}$$

For $x = 0$ we find from (41) that

$$\theta_0 = \frac{A}{p}(e^{pt} - 1) \tag{49}$$

$$= At - \frac{c_1}{L} \cdot \frac{A^2 t^2}{\underline{|2}} + \frac{c_1^2}{L^2}\frac{A^3 t^3}{\underline{|3}} - \cdots. \tag{50}$$

This shows, since A is negative, that if the thickness of ice is to increase directly as the time, the surface temperature must vary more rapidly than as a linear function of the time. For any value we wish to give A the thickness is determinate from (46).

160. Simple Solutión for Thin Ice. If we assume that the ice is thin enough so that the temperature gradient can be considered as uniform from the upper to the lower surface, we can derive at once a very simple solution; for the quantity of heat which flows upward per unit area through the ice in time dt will then be

$$-k\frac{\theta_0}{\epsilon}dt, \tag{51}$$

and this must equal the heat which is released when the ice increases in thickness by $d\epsilon$. Hence we have

$$\frac{-k_1\theta_0 dt}{\epsilon} = L\rho d\epsilon. \tag{52}$$

Integrating this and assuming that ϵ is zero when t is zero, we have

$$\epsilon^2 = \frac{-2\,\theta_0 k_1 t}{L\rho}, \tag{53}$$

which is identical with (39). This shows that the approximation involved in (39) amounts to the assumption of a uniform temperature gradient through the ice.

161. With the aid of some of his formulas Stefan calculated k for polar ice from the measured rates of ice formation at Assistance Bay, Gulf of Boothia, and other places, and found

$$k = 0.0042. \tag{54}$$

This value lies between the values attributed to Neumann (0.0057) and to Forbes (0.00223), and it is only slightly lower than that now accepted (.0052; see table of conductivity constants).

162. The fact that the conductivity of ice is considerably larger than that of water gives rise to an interesting phenomenon which has been noted by H. T. Barnes. When ice is being frozen on still water, particularly when the surface is kept very cold as by

liquid air, ice crystals grow out into the water and are found in the ice with their long axes all pointing normal to the plane of the surface. It is probable also that their conductivity is greater along this axis.

163. It may be noted in connection with the study of the formation of ice that the temperature of the surface, which, as we have seen, is the controlling factor as regards the rate of freezing, is determined by a variety of conditions; for, while in most climates and under most weather conditions this is largely dependent on the temperature of the surrounding air, in cases where the air is exceptionally clear so that an appreciable amount of radiation can take place to the outer space which is nearly at absolute zero, the surface of the ice may be considerably cooler than the air. Thus the natives of Bengal, India, make ice by exposing water in shallow earthen dishes to the clear night sky, even when the air temperature is 16° to 20° F. above the freezing point.*

164. Applications. While problems involving latent heat have been handled in the preceding chapters, the solutions have either neglected this consideration or taken account of it by some more or less rough approximation method. With the aid of the deductions of the present chapter many of these problems could now be treated rigorously, in particular such as relate to the freezing or thawing of soil. The equations would be directly applicable to this case if the thermal constants for soil were used instead of those for ice or water, and if the latent heat of fusion of ice was modified by a factor depending on the percentage of moisture in the soil.

The theory would also apply to many cases of ice formation in still water, for either natural or artificial refrigeration, while, as already noted, it has been used by Stefan in connection with polar ice.

* See Tamura, *Monthly Weather Review*, February, 1905.

PROBLEMS

1. Applying Stefan's formulas, find how long, if $\theta_0 = -15°$ C., it will take to freeze 5 cm. of ice (a) to the first approximation; and (b) to the second approximation. (3.28 hr.; 3.39 hr.)

2. Using only the first approximation of Stefan's formula, find how long it would take to thaw 5 cm. deep in a cake of ice, supposing that the water remains on top, and that the top surface of water is at $+ 15°$ C. (12.95 hr.)

3. Using Stefan's first approximation formula, find how long it would take for the soil to freeze to a depth of 1 m. if the average surface temperature is $- 10°$ C. and the soil initially at $0°$ C., and if the soil has 10% moisture. (21 days.)

4. Assume that θ_0 varies with time, so that the rate of freezing of ice is constant, and that this rate is such that 5 cm. will be frozen in the time determined in Problem 1 (a). Determine θ_0 for 1 hr., 4 hr., and 10 hr. ($- 9.5°$; $- 41°$; $- 123°$.)

5. If $C_1 = - 15°$ C. and $C_2 = + 4°$ C. in Neumann's solution, how long will it take to freeze 5 cm. of ice? (Compare with Problem 1.) (3.9 hr.)

APPENDIX A

VALUES OF THE THERMAL CONDUCTIVITY CONSTANTS

The following table has been compiled from a large variety of sources,* those results being selected which in the authors' judgment are most worthy of confidence. However, while the values for the metals (in most cases those of Jäger and Diesselhorst) are probably correct to about 1%, no such accuracy can be claimed in the case of the poorer conductors, as the disagreement between different observers is frequently 50% or even more.

When not otherwise specified ordinary temperatures are assumed.

THERMAL CONDUCTIVITY CONSTANTS IN C.G.S. UNITS

Metals	Temperature °C.	Conductivity k	Specific Heat c	Density ρ	Diffusivity $h^2\left(=\frac{k}{c\rho}\right)$
Aluminum	18	0.480	0.214	2.7[illegible]	0.826
Antimony	0	0.044	0.048	6.[illegible]	0.139
Bismuth	18	0.0194	0.0292	9.80	0.0678
Brass (yellow)	0	0.204	0.074	8.10	0.339
Cadmium	18	0.222	0.0551	8.63	0.467
Copper	18	0.918	0.0914	8.88	1.133
Gold	18	0.700	0.0308	19.21	1.182
Iron (wrought iron — also mild steel)	18	0.1436	0.1055	7.85	0.173
Iron (cast iron † and carbon (1%) steel)	18	0.108	0.114	7.82	0.121
Lead	18	0.0827	0.0308	11.32	0.237
Magnesium	0–100	0.376	0.245	1.74	0.883
Mercury	0	0.0148	0.0334	13.55	0.0327
Nickel	18	0.142	0.106	8.81	0.152
Palladium	18	0.168	0.0585	11.96	0.240
Platinum	18	0.166	0.0320	21.39	0.243
Silver	18	1.006	0.0550	10.53	1.737
Tin	18	0.155	0.0523	7.28	0.407
Zinc	18	0.263	0.0921	7.11	0.402

* For example, the Landolt-Börnstein-Meyerhoffer, and other physical tables; the articles in Winkelmann's *Handbuch*, III, Wärme (1906); papers by Jäger and Diesselhorst in the *Abh. d. phys.-tech. Reichsanstalt*, **3**, p. 269 (1900), by H. E. Patten on "Heat Transference in Soils," *Bull. No. 59*, Bureau of Soils, Department of Agriculture, and by Carl Hering in *Met. and Chem. Eng.*, December, 1911. As several constants are involved for each substance, no attempt is made at citing individual authorities.

† While some authorities class cast iron with wrought iron in this connection, the authors are inclined to accept the lower value.

THERMAL CONDUCTIVITY CONSTANTS (CONTINUED)

MATERIALS	TEMPERATURE °C.	CONDUCTIVITY k	SPECIFIC HEAT c	DENSITY ρ	DIFFUSIVITY $h^2\left(=\frac{k}{c\rho}\right)$
Air	0	0.000055	0.237	0.00129	0.179
Asbestos (loose) . . .		0.0004	0.20	0.58	0.0035
Asbestos paper		0.0006			
Brick (average fire brick)	0–800	0.0040	0.18	3.0	0.0074
Brick (average building brick — masonry) . .		0.0020			0.0050
Charcoal		0.00013			
Coal		0.0008			0.002
Concrete (cinder) . . .		0.00081		1.7	0.0032
Concrete (stone) . . .		0.0022	0.16	2.37	0.0058
Concrete (stone) . . .	20–1000	0.0027	0.21	2.3	0.0056
Concrete (very light slag)		0.00053		0.55	0.006
Cork (ground)		0.00012	.48	.15	0.0017
Cotton (loose)		0.00015			
Ebonite		0.00040	0.34	1.15	0.0010
Glass (ordinary) . . .		0.0024	0.161	2.60	0.0057
Granite		0.0081	0.196	2.66	0.0155
Ice		0.0052	0.502	0.92	0.0112
Limestone		0.0050	0.217	2.5	0.0092
Magnesium carbonate (85% — steam-pipe covering)		0.00017			
Marble (white)		0.0050			0.0090
Mineral wool		0.0001			
Paraffin		0.00061	0.69	0.90	0.00098
Rock material (average for the earth — Kelvin)		0.0042			0.0118
Rock material (an average for crustal rocks, sometimes used) . .		0.0045	.25	2.8	0.0064
Sandstone		0.0050			0.0133
Snow (fresh		0.0003	0.50	0.18	0.0033
Soil (clay or sand, slightly damp, average) . .		0.0037	0.45	1.65	0.0049 *
Soil (very dry)		0.00088			0.0031
Water		0.00143	1.00	1.00	0.00143
Wood (pine — across grain)		0.00009	0.33	0.4	0.00068
Wood (pine, with grain)		0.00030	0.33	0.4	0.0023
Wool (sheep's, loose) .		0.00014			

* This is possibly somewhat low as a general average.

VALUES OF THE EMISSIVITY FACTOR E FOR AIR AT ONE ATMOSPHERE PRESSURE

Excess Temperature		E
0°–50° C.	Polished surface	.00018–.00023
	Blackened surface	.00025–.00033
	(In a vacuum, ½ to ⅓ of these values)	
300°	Platinum wire	.00088
1000°	Platinum wire	.0020

APPENDIX B

INDEFINITE INTEGRALS

$$\int u\,dv = uv - \int v\,du.$$

$$\int e^{ax}dx = \frac{e^{ax}}{a}.$$

$$\int \frac{dx}{x} = \log x.$$

$$\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1).$$

$$\int x^m dx = \frac{x^{m+1}}{m+1}, \text{ if } m \neq -1.$$

$$\int a^{bx}dx = \frac{a^{bx}}{b\log a}.$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

$$\int (x^2 \pm a^2)^{\frac{1}{2}}dx = \frac{1}{2}\left[x\sqrt{x^2\pm a^2} \pm a^2\log\left(x+\sqrt{x^2\pm a^2}\right)\right].$$

$$\int (a^2-x^2)^{\frac{1}{2}}dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right].$$

$$\int \sec^2 x\,dx = \tan x.$$

$$\int x^2\sin x\,dx = 2x\sin x - (x^2-2)\cos x.$$

$$\int \tan x\,dx = -\log\cos x.$$

$$\int x^2\cos x\,dx = 2x\cos x + (x^2-2)\sin x.$$

$$\int x\sin ax\,dx = \frac{1}{a^2}[\sin ax - ax\cos ax].$$

$$\int x\cos ax\,dx = \frac{1}{a^2}[\cos ax + ax\sin ax].$$

$$\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \quad a \neq b.$$

$$\int \sin ax\cos bx\,dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}.$$

$$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}.$$

$$\int \sin^2 ax\,dx = \frac{1}{2a}(ax - \sin ax\cos ax).$$

$$\int \cos^2 ax\,dx = \frac{1}{2a}(ax + \sin ax\cos ax).$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx).$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx).$$

APPENDIX C

DEFINITE INTEGRALS

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx.$$

$$\int_0^\infty \frac{\sin^2 x dx}{x^2} = \frac{\pi}{2}.$$

$$\int_0^\infty \frac{\sin ax dx}{x} = \frac{\pi}{2}, \text{ if } a > 0; \ 0, \text{ if } a = 0; \ -\frac{\pi}{2}, \text{ if } a < 0.$$

$$\int_0^\infty \frac{\sin x \cos ax dx}{x} = 0, \text{ if } a < -1 \text{ or } > 1; \ \frac{\pi}{4}, \text{ if } a = -1 \text{ or } +1;$$
$$\frac{\pi}{2}, \text{ if } 1 > a > -1.$$

$$\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

$$\int_0^\pi \sin ax \sin bx dx = \int_0^\pi \cos ax \cos bx dx = 0, \text{ if } a \neq b.$$

$$\int_0^\pi \sin^2 ax dx = \int_0^\pi \cos^2 ax dx = \frac{\pi}{2}.$$

$$\int_0^\infty e^{-a^2x^2} dx = \frac{1}{2a}\sqrt{\pi}.$$

$$\int_0^\infty x^n e^{-ax} dx = \frac{\lfloor n}{a^{n+1}}.$$

$$\int_0^\infty e^{-a^2x^2} \cos bx dx = \frac{\sqrt{\pi}}{2a} e^{-\frac{b^2}{4a^2}}, \text{ if } a > 0.$$

$$\int_{-\infty}^{+\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

APPENDIX D

TABLE OF VALUES* OF THE "PROBABILITY INTEGRAL"

$$\Theta(q) = \frac{2}{\sqrt{\pi}} \int_0^q e^{-\beta^2}\, d\beta \left[= \frac{2}{\sqrt{\pi}} \int_{-q}^{0} e^{-\beta^2}\, d\beta \right]$$

FOR VALUES OF THE ARGUMENT q

q	$\Theta(q)$	q	$\Theta(q)$	q	$\Theta(q)$	q	$\Theta(q)$
0.00	0.00000	0.66	0.64938	1.30	0.93401	1.94	0.99392
0.02	0.02256	0.68	0.66378	1.32	0.93807	1.96	0.99443
0.04	0.04511	0.70	0.67780	1.34	0.94191	1.98	0.99489
0.06	0.06762	0.72	0.69143	1.36	0.94556	2.00	0.99532
0.08	0.09008	0.74	0.70468	1.38	0.94902	2.05	0.99626
0.10	0.11246	0.76	0.71754	1.40	0.95229	2.10	0.99702
0.12	0.13476	0.78	0.73001	1.42	0.95538	2.15	0.99764
0.14	0.15695	0.80	0.74210	1.44	0.95830	2.20	0.99814
0.16	0.17901	0.82	0.75381	1.46	0.96105	2.25	0.99854
0.18	0.20094	0.84	0.76514	1.48	0.96365	2.30	0.99886
0.20	0.22270	0.86	0.77610	1.50	0.96611	2.35	0.9991107
0.22	0.24430	0.88	0.78669	1.52	0.96841	2.40	0.9993115
0.24	0.26570	0.90	0.79691	1.54	0.97059	2.50	0.9995930
0.26	0.28690	0.92	0.80677	1.56	0.97263	2.60	0.9997640
0.28	0.30788	0.94	0.81627	1.58	0.97455	2.70	0.9998657
0.30	0.32863	0.96	0.82542	1.60	0.97635	2.80	0.9999250
0.32	0.34913	0.98	0.83423	1.62	0.97804	2.90	0.9999589
0.34	0.36936	1.00	0.84270	1.64	0.97962	3.00	0.9999779
0.36	0.38933	1.02	0.85084	1.66	0.98110	3.10	0.9999884
0.38	0.40901	1.04	0.85865	1.68	0.98249	3.20	0.9999940
0.40	0.42839	1.06	0.86614	1.70	0.98379	3.30	0.9999969
0.42	0.44747	1.08	0.87333	1.72	0.98500	3.40	0.9999985
0.44	0.46623	1.10	0.88020	1.74	0.98613	3.50	0.99999925691
0.46	0.48466	1.12	0.88679	1.76	0.98719	3.60	0.99999964414
0.48	0.50275	1.14	0.89308	1.78	0.98817	3.70	0.99999983285
0.50	0.52050	1.16	0.89910	1.80	0.98909	3.80	0.99999992300
0.52	0.53790	1.18	0.90484	1.82	0.98994	3.90	0.99999996521
0.54	0.55494	1.20	0.91031	1.84	0.99074	4.00	0.99999998458
0.56	0.57162	1.22	0.91553	1.86	0.99147	4.20	0.99999999714
0.58	0.58792	1.24	0.92051	1.88	0.99216	4.40	0.99999999951
0.60	0.60386	1.26	0.92524	1.90	0.99279	4.60	0.99999999992
0.62	0.61941	1.28	0.92973	1.92	0.99338	4.80	0.99999999999
0.64	0.63459					∞	1.00

* Wellisch, *Theorie und Praxis der Ausgleichsrechnung*, p. 257.

APPENDIX E

VALUES OF e^{-x}

These may best be taken at once from an ordinary logarithm table as values of $\frac{1}{10^{.4343\,x}}$, but the following abbreviated table may prove of occasional convenience.

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
0.00	1.00000	1.00	0.36788	2.50	0.08209	5.00	0.00674
0.20	0.81873	1.30	0.27253	3.00	0.04979	6.00	0.00248
0.40	0.67032	1.60	0.20109	3.50	0.03020	7.00	0.00091
0.60	0.54880	1.80	0.16530	4.00	0.01832	8.00	0.00034
0.80	0.44933	2.00	0.13533	4.50	0.01111	9.00	0.00012

APPENDIX F

MISCELLANEOUS FORMULAS

$e = 2.71828.$

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots. \qquad [x^2 < \infty.]$$

$$\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots. \qquad [x^2 < 1.]$$

$$\log_e x = \log_a x \cdot \log_e a = 2.3026 \log_{10} x.$$

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots. \qquad [x^2 < \infty.]$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots. \qquad [x^2 < \infty.]$$

$$e^{ix} = \cos x + i \sin x.$$

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

$$\frac{d}{db}\int_a^b f(x)\,dx = f(b).$$

$$\frac{d}{da}\int_a^b f(x)\,dx = -f(a).$$

$$\frac{d}{dc}\int_a^b f(x, c)\,dx = \int_a^b \frac{\partial}{\partial c} f(x, c)\,dx + f(b, c)\frac{db}{dc} - f(a, c)\frac{da}{dc}.$$

$$\int_a^b f(x)\,dx = (b - a)f(\alpha), \text{ where } a < \alpha < b.$$

$$f(x + h) = f(x) + hf'(x + \alpha h), \text{ where } 0 < \alpha < 1.$$

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